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On Empirical System Gramians

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State-space realizations of input-output systems or control systems are a widely used class of models in engineering, physics, chemistry and biology. For the qualitative and quantitative classification of such systems, the system-theoretic properties of reachability and observability are essential, which are encoded in so-called system Gramian matrices. For linear systems these Gramians are computed as solutions to matrix equations, for nonlinear or parametric systems the data-driven empirical system Gramians approximate the actual system Gramians. These empirical Gramians have manifold applications, for example in model reduction or decentralized control of nonlinear systems, as well as sensitivity analysis, parameter identification and combined state and parameter reduction of parametric systems. Here, we demonstrate that empirical system Gramians are also useful for linear but hyperbolic input-output systems.

1 Empirical System Gramians

System Gramian matrices were introduced in [1] to quantify reachability and observability of linear control systems. Linear (time-invariant) systems map an input $u : \mathbb{R} \to \mathbb{R}^M$ via the state $x : \mathbb{R} \to \mathbb{R}^N$ of a linear dynamical system to an output $y : \mathbb{R} \to \mathbb{R}^Q$:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t). \tag{1}$$

The system Gramians: Reachability Gramian W_R (for linear time-invariant systems the controllability Gramian W_C is equivalent), observability Gramian W_O and cross Gramian W_X (encoding minimality based on joint reachability and observability), are then defined as:

$$W_R := \int_0^\infty e^{At} BB^\intercal e^{A^\intercal t} dt, \quad W_O := \int_0^\infty e^{A^\intercal t} C^\intercal C e^{At} dt, \quad W_X := \int_0^\infty e^{At} BC e^{At} dt, \tag{2}$$

and their rank, eigenvalues, singular values, trace, determinant, norms, signature or more generally spectral properties encode (reachability, observability and minimality) information about the associated input-output system (1). These linear system Gramians can be computed (also as low-rank approximation) as solution to the matrix equations:

$$AW_R + W_R A^\intercal = -BB^\intercal, \quad A^\intercal W_O + W_O A = -C^\intercal C, \quad AW_X + W_X A = -BC. \tag{3}$$

For nonlinear systems of the general form,

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = g(x(t), u(t)), \tag{4}$$

the system Gramians cannot be expressed as for linear systems (2) since the vector field f and output functional g do not have the (A, B, C) form, and hence cannot be computed by (3). And typically, a plain linearization of the system (4) to the form (1) is only suitable for weakly nonlinear systems. The empirical system Gramian matrices resolve this problem in a data-driven manner utilizing numerical quadrature. The factors in the integrands of the linear system Gramians (2) are impulse response and adjoint impulse response of the system (1). A numerical approximation, for example by a Runge-Kutta method, of these impulse responses based on simulated state trajectories $\tilde{x}^m(t)$ and output trajectories $\tilde{y}^n(t)$ is given by:

$$X(t) := \begin{bmatrix} \tilde{x}^1(t) & \dots & \tilde{x}^M(t) \end{bmatrix}, \quad Y(t) := \begin{bmatrix} \tilde{y}^1(t) & \dots & \tilde{y}^N(t) \end{bmatrix}, \tag{5}$$

with the indices m and n signifying a perturbation of the m-th steady-state input or the n-th steady-state component of the system (4). This leads to the (simplified) empirical reachability Gramians $\widetilde{W}_R$, empirical observability Gramian $\widetilde{W}_O$ and empirical cross Gramian $\widetilde{W}_X$:

$$\widetilde{W}_R := \int_0^\infty X(t)X(t)^\intercal dt, \quad \widetilde{W}_O := \int_0^\infty Y(t)^\intercal Y(t) dt, \quad \widetilde{W}_X := \int_0^\infty X(t)Y(t) dt. \tag{6}$$

Notably, these empirical system Gramians (6) are equivalent to the linear system Gramians (2) up to numerical error, if constructed using impulse input for linear systems (1). Hence, empirical Gramians can be seen as a generalization of linear system Gramians, i.e.: $X(t) = e^{At}B$, $Y(t) = Ce^{At}$. For an overview and detailed description of empirical Gramians and their computation see [2].

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2 Linear Advection Example

While empirical system Gramians are designed for nonlinear and parametric systems, a data-driven computation can be advantageous even for linear systems (1). This is demonstrated using a principal application of (empirical) system Gramians: Model reduction. Gramian-based model reduction computes (Petrov-)Galerkin projection operators

$$U : \mathbb{R}^n \to \mathbb{R}^N, \quad V : \mathbb{R}^N \to \mathbb{R}^n, \quad n < N, \tag{7}$$

from system Gramian matrices, i.e. [3]. Some examples are: Proper orthogonal decomposition (POD), balanced truncation (BT), and the dominant subspace projection model reduction (DSPMR) method. Applying U and V to the system matrices yields the reduced order system (A_r, B_r, C_r) with:

$$A_r := V \circ A \circ U, \quad B_r := V \circ B, \quad C_r := C \circ U. \tag{8}$$

For a controlled one-dimensional linear advection equation – a first order hyperbolic partial differential equation – on a unit domain, with (left-hand) boundary control and (right-hand) boundary observer [4, Linear Advection], we have

$$\frac{\partial z}{\partial t} = -a \frac{\partial z}{\partial x}, \quad z(0,t) = u(t), \quad y(t) = z(1,t),$$

with transport velocity $a = 1$. A finite difference upwind scheme spatial discretization yields a system (1) of order $N = 1000$. The model reduction techniques POD, BT and DSPMR, based on computation via (3) and (6) are compared in the resulting relative output model reduction error $\varepsilon_{\text{MOR}} = \|y - y_r\|_{L_2} / \|y\|_{L_2}$ for different reduced orders. In Figure 1, POD, BT and (cross-Gramian-based) DSPMR [5], computed via matrix equations and empirically, are compared, whereas the empirical system Gramians are computed using the empirical Gramian framework `emgr` [6]. The reduced order models (ROMs) are tested using a smooth input signal which is not included in the training set for the empirical Gramians, encompassing step functions, and the first order Euler's method with suitable time discretization is used to generate the trajectory data. Overall, the system Gramians computed from matrix equations (3) produce less accurate ROMs than their empirical

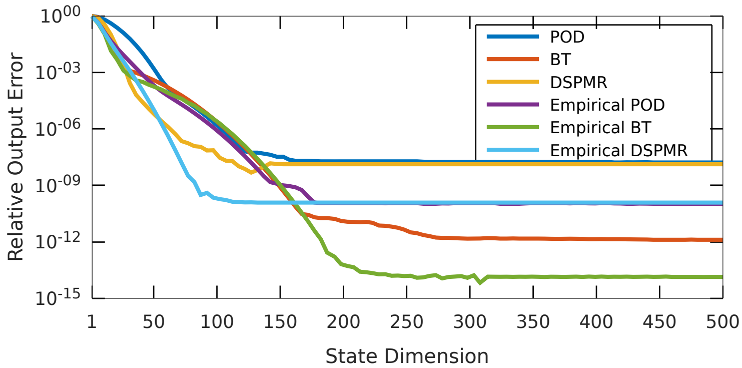

Fig. 1: Comparison of relative output model reduction error for increasing reduced order of POD, BT, and DSPMR methods utilizing matrix-equation-based and empirical system Gramians.

counterparts. The smallest ROM with suitably low error is given by the empirical DSPMR method ($n = 90$), while a suitably small ROM with the lowest error is given by empirical balanced truncation ($n = 215$). These numerical experiments indicate that even for linear systems with complexities such as hyperbolicity, empirical system Gramians can capture more information. Future work will investigate specialized empirical Gramians for linear and nonlinear hyperbolic input-output systems.

Code Availability Section Source code for the numerical experiment: `http://www.runmycode.org/companion/view/3477`

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