Westfälische Wilhelms-Universität Münster

COMBINED REDUCTION

Combined State and Parameter Reduction for Large-Scale Bayesian Inverse Problems

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About

Combined Reduction

Combined reduction enables the concurrent reduction of state and parameter space dimensions, which accelerates the integration. In a bayesian inversion setting with undetermined posterior parameter distributions, the estimation duration is significantly shortened.

Method

Empirical Cross Gramian Based

Empirical gramians, introduced in [1], allow to compute gramian matrices for nonlinear systems that match the classic controllability, observability and cross gramian for linear systems. The cross gramian carries both controllability and observability information, thus permits to avoid a costly balancing procedure.

Greedy Optimization Based

Computing a state and parameter projection iteratively by optimization was established in [2]. The associated greedy objective function minimizes the maximum error between full and reduced model yielding a parameter projection basis vector from which a state basis vector is generated.

New

Joint Gramian

Extending the systems states by constant parameter states and the systems inputs to act upon those, results in an augmented system. The computation of the cross gramian of such an augmented system gives the joint gramian (see [3]), which conveys identifiability information on states and parameters.

Data-Driven & Trust-Region

Utilizing the experimental data, instead of generating simulations, reduces the cost inside each iterations optimization. Furthermore using a trust-region approach, which starts with a single parameter and increments the parameter space dimension after each iteration, significantly lowers each optimizations duration.

Algorithm

Joint Gramian Direct Truncation

1. Augment system: $F = (f(x,u,\theta); u)$; $G = (g(x,u,\theta); \theta)$
2. Compute empirical joint gramian W_J
3. Extract cross gramian $W_x = \mathit{upperleftblock}(W_J)$
4. Extract cross identifiability gramian $W_{\bar{i}} = \mathit{schurcomplement}(W_J)$
5. Take SVD of gramians: $W_X = U_X D_X U_X^T$; $W_{\bar{i}} = U_{\bar{i}} D_{\bar{i}} U_{\bar{i}}^T$
6. Truncate states i with $D_{X i,i} < s$ and parameters j with $D_{\bar{i} j,j} < p$.
7. Optimize reduced system

Trust-Region Greedy Optimization

1. Initialize parameter projection P and state projection X
2. Do:
   - 2.1 Minimize $\max(\|y_r - y_d\|_2) + \alpha\|\theta_p\|_S$ over parameters p
   - 2.2 Append parameter projection $P = \text{orth}([P,p])$
   - 2.3 Generate snapshot from current reduced system $x_r(p)$
   - 2.4 Append state projection by averaged snapshot $X = \text{orth}([X, \bar{x}_r(p)])$
3. Optimize reduced system

Features

gramian_comred.m

- Linear and nonlinear systems
- Square and (currently only) symmetric systems
- Flexible snapshot configuration
- Utilizes empirical gramian framework (emgr)
- Lengthy offline times

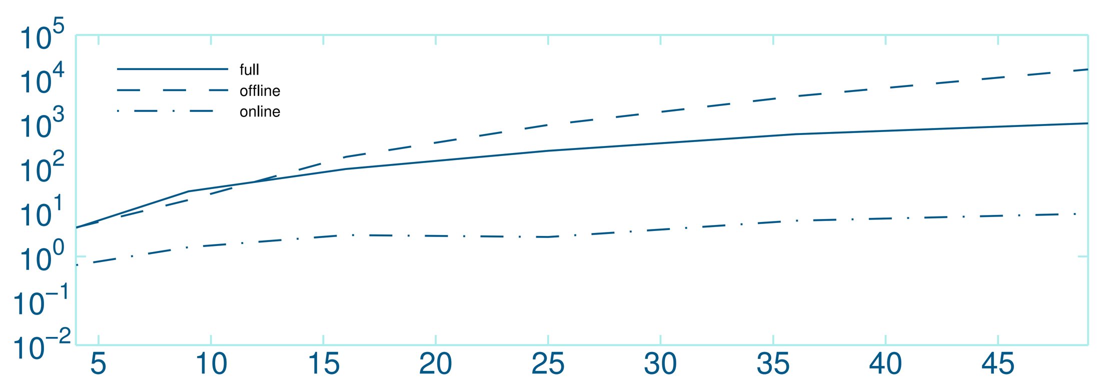
greedy_comred.m

- (Currently only) Linear systems
- Non-symmetric and non-square systems
- Enables extreme-scale system reduction
- No dependencies
- Very short offline times

Comparison

Full vs Gramian

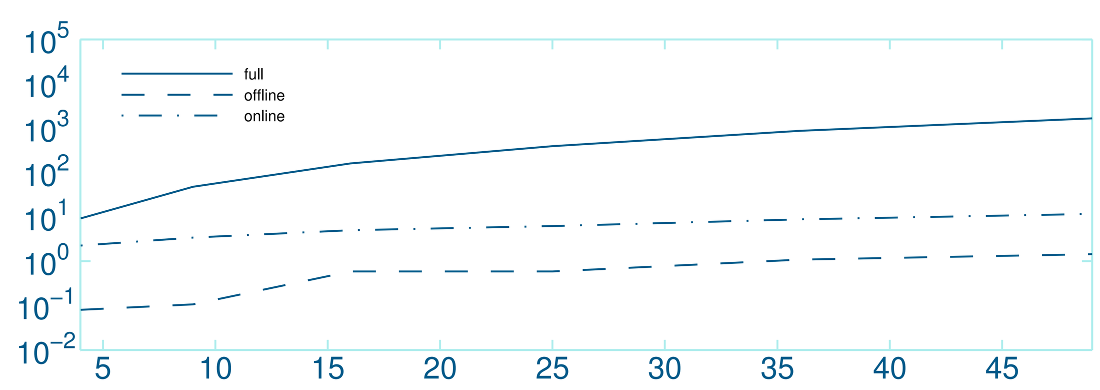
Full order optimization duration compared to offline and online durations for different linear system state dimensions:



The short online times justify a long onetime offline duration.

Full vs Greedy

Full order optimization duration compared to offline and online durations for different linear system state dimensions:



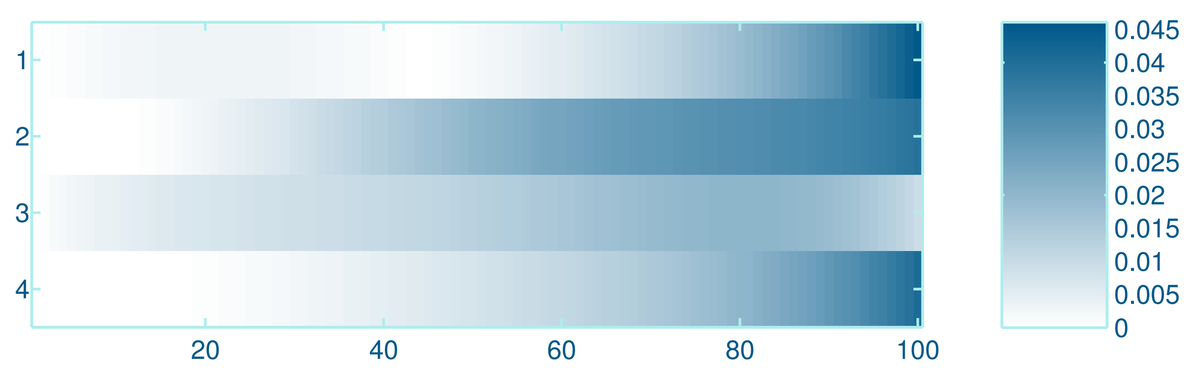
The short offline times allow reduction of models even on extreme scales.

Example

Large-Scale Nonlinear System

$$\Sigma_1 = \begin{cases} \dot{x} = A\tanh(x) + Bu \\ y = Cx \end{cases}$$

- 16 States
- 256 Parameters
- 4 Inputs
- 4 Outputs
- 4 Reduced States
- 16 Reduced Parameters
- L2-Error: 0.013
- Total Time: 77.2s
- Offline Time: 75.4s
- Online Time: 1.8s

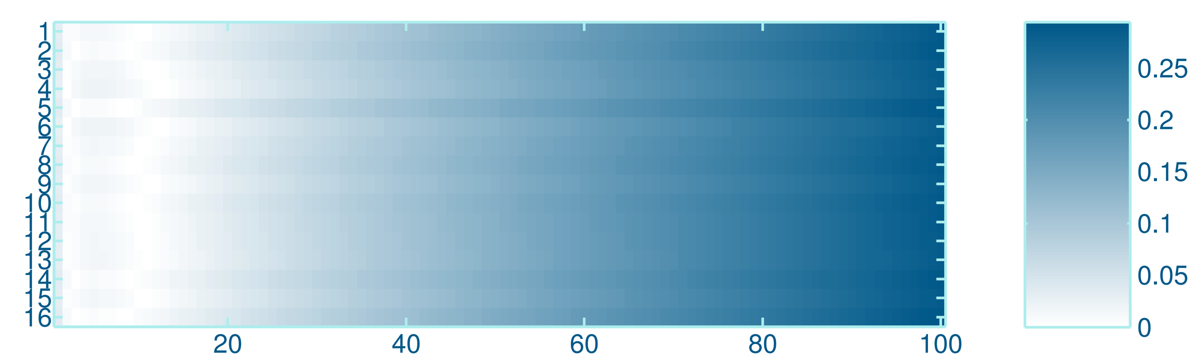


Relative error in outputs of reduced Σ_1.

Extreme-Scale Linear System

$$\Sigma_2 = \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

- 256 States
- 65536 Parameters
- 16 Inputs
- 16 Outputs
- 16 Reduced States
- 256 Reduced Parameters
- L2-Error: 0.018
- Total Time: 179.0s
- Offline Time: 12.7s
- Online Time: 176.3s



Relative error in outputs of reduced Σ_2.

Get

Read Me

[1] S.Lall, J.E. Marsden, and S. Glavaski. Empirical model reduction of controlled nonlinear systems. *Proceedings of the IFAC World Congress*, F:473–478, 1999.
[2] C. Lieberman, K. Willcox and O. Ghattas. Parameter and state model reduction for large-scale statistical inverse problems. *SIAM Journal on Scientific Computing*, 32(5): 2523-2542, 2010.
[3] C. Himpe and M. Ohlberger. Cross-Gramian Based Combined State and Parameter Reduction. *Preprint*, arXiv:1302.0634 (2013).

- Source code available at: `http://j.mp/modred` under open source license and compatible with MATLAB and OCTAVE