WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER

APPLIED MATHEMATICS MÜNSTER

emgr

Empirical Gramian Framework

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Abstract

Gramian-based model reduction is a well established method for linear state-space systems. Beyond linear systems, empirical gramian matrices expand the scope of system-theoretic methods to nonlinear systems. Additionally, empirical gramians can also be used for system identification and combined state and parameter reduction.
The **em**pirical **gr**amian framework emgr is a compact Matlab toolbox enabling nonlinear model order reduction.

http://gramian.de

Download Demos Documentation

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Scope

- Model Reduction
- Parametric Model Order Reduction
- Robust Reduction
- Sensitivity Analysis
- Parameter Identification
- Parameter Reduction
- Combined State and Parameter Reduction !
- Decentralized Control
- Nonlinearity Quantification
- Uncertainty Quantification

Application

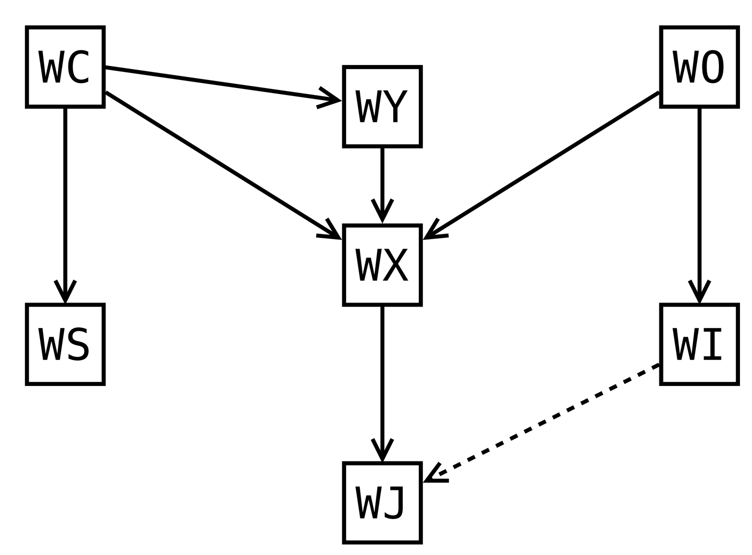
- Linear Systems: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$
- Nonlinear System: $\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases}$
- Second Order Systems: $\begin{cases} \ddot{x}(t) = f(\dot{x}(t), x(t), u(t)) \\ y(t) = g(\dot{x}(t), x(t), u(t)) \end{cases}$
- Parametrized Systems: $\begin{cases} \dot{x}(t) = f(x(t), u(t), \theta) \\ y(t) = g(x(t), u(t), \theta) \end{cases}$
- Discretized PDEs

Mathematical Background

- Projection-Based Model Reduction: $\begin{cases} \dot{x}_r(t) = Vf(Ux_r(t), u(t), \theta) \\ y_r(t) = g(Ux_r(t), u(t), \theta) \end{cases}, \quad \|y - y_r\| \ll 1, \quad VU = \mathbb{1}$
- Impulse Response: $h(t > 0) = Ce^{At}B \Rightarrow y(t) = (h * u)(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau =: S(u)(t)$
- Hankel Singular Values: $F(u) := u(-t) \to S \circ F = \int_0^\infty Ce^{A(t-\tau)}Bu(-\tau)d\tau =: H(u)(t) \Rightarrow \|H\|_* < \infty$
- Controllability & Observability: $\mathcal{C}(u) := \int_0^\infty e^{At}Bu(-t)dt, \quad \mathcal{O}(x) := Ce^{At}x, \quad H = \mathcal{O}\mathcal{C}$
- System Gramians: $W_C := \mathcal{C}\mathcal{C}^T, \quad W_O := \mathcal{O}^T\mathcal{O}, \quad W_X := \mathcal{C}\mathcal{O}$
- Balancing / Approximate Balancing: $\sigma(H) = \sqrt{\lambda(W_C W_O)} \overset{sym}{=} |\lambda(W_X)| \approx \sigma(W_X)$
- Balanced Truncation: $W_C^{\frac{1}{2}} W_O^{\frac{1}{2}} \overset{SVD}{=} UDV \to U = (U_1, U_2), \quad V = (V_1, V_2)^T$
- Direct Truncation: $W_X \overset{SVD}{=} UDV \to U = (U_1, U_2), \quad V := U^T$
- Parameter Augmented System: $\begin{pmatrix} \dot{x}(t) \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} f(x(t), u(t), \theta) \\ 0 \end{pmatrix}, \quad y(t) = g(x(t), u(t), \theta)$

$\mathrm{tr}(\mathcal{O}\mathcal{C}) = \mathrm{tr}(\mathcal{C}\mathcal{O})$

Trace property for the cross gramian and Hankel operator.



Relation between system gramians.

Empirical Gramians

- Empirical Controllability Gramian $W_C := \int_0^\infty e^{At}BB^Te^{A^Tt}dt = \sum_i \int_0^\infty X_i X_i^T dt \approx \sum_i \frac{1}{h} \sum_t^T X_i X_i^T$
- Empirical Observability Gramian $W_O := \int_0^\infty e^{A^Tt}C^TCe^{At}dt = \sum_i \int_0^\infty Y_i^T Y_i dt \approx \sum_i \frac{1}{h} \sum_t^T Y_i^T Y_i$
- Empirical Cross Gramian $W_X := \int_0^\infty e^{At}BCe^{At}dt = \sum_i \int_0^\infty X_i Y_i dt \approx \sum_i \frac{1}{h} \sum_t^T X_i Y_i$
- Empirical Linear Cross Gramian $W_Y := \int_0^\infty e^{At}B(e^{A^Tt}C^T)^T dt = \sum_i \int_0^\infty X_i Z_i^T dt \approx \sum_i \frac{1}{h} \sum_t^T X_i Z_i^T$
- Empirical Sensitivity Gramian $W_S := \mathrm{diag}(\mathrm{tr}(W_C(u = \theta_i)))$
- Empirical Identifiability Gramian $\hat{W}_O := \begin{pmatrix} W_O & W_M \\ W_M^T & W_P \end{pmatrix} \Rightarrow W_I = W_P - W_M^T W_O^{-1} W_M \approx W_P$
- Empirical Joint Gramian $W_J := \hat{W}_X = \begin{pmatrix} W_X & W_M \\ 0 & 0 \end{pmatrix} \Rightarrow W_{\ddot{I}} := -\frac{1}{2} W_M^T (W_X + W_X^T)^{-1} W_M$

Optional

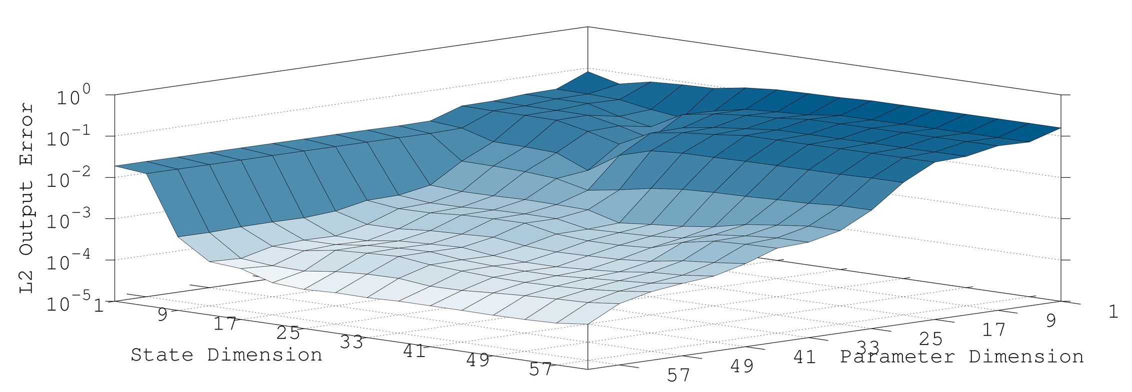
- Empirical Covariance Matrices
- Non-Symmetric Cross Gramian !
- Data-Driven Gramians
- Robust Gramians
- Global Parametric Gramians

Algorithmical

- Improved Runge-Kutta
- Generalized Transpositions
- Fast Approximate Inverse
- Eigenvalue Balancing
- Re-Orthogonalized Lanczos SVD

Gramian Type	State Reduction	Parameter Reduction	Combined Reduction	Sensitivity Analysis
WC	(✔)	✘	✘	✘
WO	(✔)	✘	✘	✘
WC + WO	✔	✘	✘	✘
WX / WY	✔	✘	✘	(✔)
WS	(✔)	✔	✘	✔
WI	(✔)	✔	(✔)	✔
WJ	(✔)	✔	✔	✔

Empirical gramian application matrix.



Combined state and parameter reduction of a hyperbolic network model.

Conceptual

- Open Source (BSD 2-Clause-License)
- Uniform Interface
- Designed like an App
- Default Solver & Custom Solver Interface
- Configurable

Technical

- Only Simple Matrix & Vector Operations
- Vectorized
- Implicit Parallelization
- Explicit Parallelization Hints
- Compatible with Matlab, Octave, FreeMat

Under Construction

- Adaptive Improved Runge-Kutta Solver
- Gramian Computation on GPU Accelerators by Automatic Offloading
- Zero-Copy / hUMA Utilization
- Distributed Memory Empirical Cross Gramian Assembly & Distributed Memory SVD
- Python (NumPy / SciPy) Variant

Version 3.0

More info? : @modelreduction

Read Me

- B. Moore, Principal component analysis in linear systems: Controllability, observability, and model reduction, IEEE Transactions on Automatic Control, 26(1): 17–32, 1981.
- S. Lall, J.E. Marsden and S. Glavaski, Empirical model reduction of controlled nonlinear systems. *Proceedings of the IFAC World Congress*, F:473–478, 1999.
- C. Himpe and M. Ohlberger, A Unified Software Framework for Empirical Gramians, Journal of Mathematics, 2013: 1–6, 2013.
- C. Himpe and M. Ohlberger, Model Reduction for Complex Hyperbolic Networks, Proceedings of the European Control Conference'14, 2739–2743, 2014.
- C. Himpe and M. Ohlberger, Cross-Gramian-Based Combined State and Parameter Reduction for Large-Scale Control Systems, Mathematical Problems in Engineering, 2014: 1–13, 2014.
- C. Himpe and M. Ohlberger, A Note on the Cross Gramian for Non-Symmetric Systems, Submitted, arXiv:1501.05519, 2015.