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Optimization-Based Combined Reduction for Nonlinear Systems

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Abstract

In applications requiring model-constrained optimization, model reduction may be indispensable to facilitate an acceptable timescale for the solution. For models with high-dimensional state- and also high-dimensional parameter-spaces the optimization is impeded twice. First, due to the high-dimensional parameter-space many solutions for varying locations in the parameter-space are usually required, second, each of these solutions is costly due to the high-dimensional state-space. A combined state- and parameter-space reduction as proposed in [1] can adress these issues. This combined reduction relies on a greedy sampling of the parameter-space to iteratively assemble a low-dimensional parameter base and a POD-based reduction at locations of the parameter base components. Yet, the greedy algorithm still requires the sampling in the high-dimensional parameter-space. An extensions to this algorithm is proposed in [2], which uses a Monte-Carlo approach to select low-dimensional "hyper"-bases for the parameter-space over which the greedy sampling is performed. And since this combined reduction only requires solutions of the associated model, it is generally applicable also to nonlinear system, which will be demonstrated.

github.com/gramian/optmor
open-source, compatible with Matlab & Octave

Model Properties

- Nonlinear control system model,
- assumed to be asymptotically stable,
- and a nonlinear parameter mapping $\theta \mapsto [f, g]$,
- high-dimensional state-space $\dim(x(t)) \gg 1$,
- high-dimensional parameter-space $\dim(\theta) \gg 1$,
- low-dimensional input-space $\dim(u(t)) \ll \dim(x(t))$,
- low-dimensional output-space $\dim(y(t)) \ll \dim(x(t))$.

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), \theta), \\ y(t) &= g(x(t), u(t), \theta), \\ x(0) &= x_0, \\ u(t) &\in L_2 \end{aligned}$$

Parameter-Space Reduction

$$\theta_{I+1} = \operatorname*{arg\,max}_{\theta \in \Theta, \|\theta\|=1} \|y(\theta) - y(\Pi_I \Pi_I^T \theta)\|_2^2,$$

subject to:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), \theta), \\ y(t) &= g(x(t), u(t), \theta), \\ x(0) &= x_0, \\ \Pi_I^T \Pi_I &= \mathbb{1}, \end{aligned}$$

- Iterative procedure, each iteration extends reduced basis incrementally,
- based on best approximating subspace,
- using (adaptive) greedy sampling strategy [3],
- selecting location in the parameter-space that maximizes the error between full and reduced order model (ROM),
- leads to model-constrained optimization problem,
- with regularization handling parameter-space constraints [4], here Tikhonov regularization;
- numerical computation by negation of cost-functional resulting in a minimzation problem.

State-Space Reduction

- Extract dominant mode(s) from current iterations selected parameter,
- using energy-based methods, either:
- input-to-state mapping: proper orthogonal decomposition (POD) [5],
- state-to-output mapping: Hessian-based reduction [6],
- input-to-output mapping: goal-oriented reduction [7],
- state-space reduction depends on parameter-space basis;
- for nonlinear systems POD is the most accessible method,

$$U_I = \operatorname*{arg\,min}_{U^T U = \mathbb{1}} \|r(U)\|_2^2$$

$$r = \begin{cases} x(u) - x_r(u; U) \\ y(x_0) - y_r(x_{r,0}; U) \\ y(u) - y_r(u; U) \end{cases}$$

Optimization-Based Combined Reduction

$\theta_0 \longleftarrow \bar{\theta}$
$\Pi_0 \longleftarrow \theta_0$
$\bar{x}_0 \longleftarrow Pod_1(x(\theta_0))$
$U_0 \longleftarrow \bar{x}(\theta_0)$
for $I = 1 : R$ **do**
  $\theta_{I+1} \longleftarrow \arg\min -J(\theta; U_I, \Pi_I) + \mathcal{R}_2(\theta; \Pi_I)$
  $\Pi_{I+1} \longleftarrow orth(\Pi_I, \theta_I)$
  $U_{I+1} \longleftarrow orth(U_I, \bar{x}(\theta_I))$

Combined Reduction

- Fuses greedy-algorithm-based parameter-space reduction with the POD-based state-space reduction,
- initialized by nominal (prior) parameter,
- solves each iteration the greedy regularized optimization problem,
- incorporates new parameter and state base components by orthogonalization,
- iterates until a certain base size or error criteria is met,
- assembles parameter and state Galerkin projections; algorithm based on [1].

Enhancements

- Inverse problems naturally contain observed / measured data,
- by a data-driven regularization operator [2],
- this improves ROM accuracy, but fixes ROM to data-set.
- Greedy sampling still requires high-dimensional optimization,
- selecting a random low-dimensional random hyper-base each iteration,
- leads to the Monte-Carlo base extension [2] which reduces offline times.
- Both enhancements complement each other.

$$\mathcal{R}_d(\theta) := \|y_d - y_r(\theta)\|_2^2$$

$$\rightarrow \theta_I = \operatorname*{arg\,max}_{\theta \in \Theta} \|y(\theta) - y_r(\theta_r)\|_2^2 - \beta_2 \mathcal{R}_2(\theta) - \beta_d \mathcal{R}_d(\theta)$$

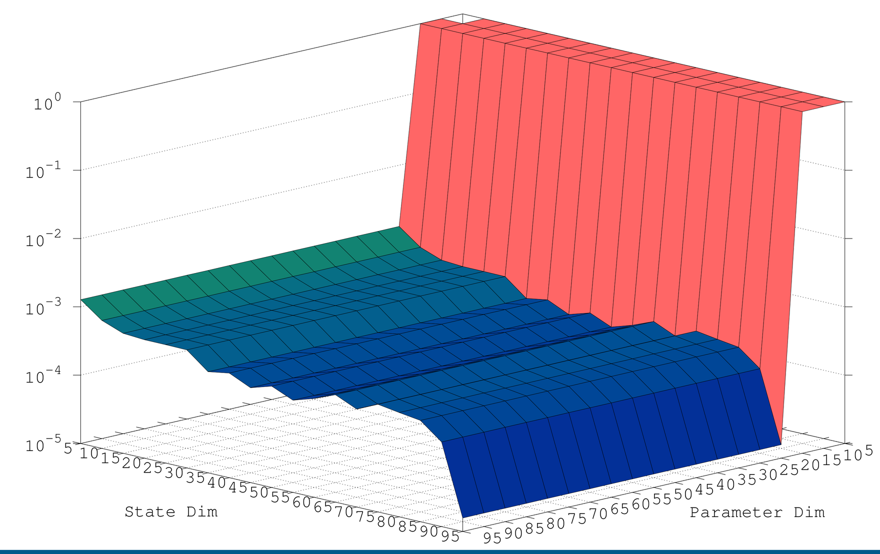
$$\theta \approx \tilde{\theta} := \sum_{i \ll \dim(\theta)} a_i r_i, \quad r_i \leftarrow \mathcal{U}_{[0,1]}$$

$$\rightarrow \theta_I = \operatorname*{arg\,max}_{\theta \in \Theta} \|y(\tilde{\theta}_r) - y_r(\tilde{\theta}_r)\|_2^2 - \beta_2 \mathcal{R}_2(\tilde{\theta})$$

Numerical Test

$$\begin{aligned} \dot{x}(t) &= A \tanh(K(\theta) x(t)) + Bu(t), \\ y(t) &= Cx(t), \\ x(0) &= 0, \\ u(t) &= \delta(t) \end{aligned}$$

- SISO system,
- hyperbolic network model,
- random, but stable network,
- state-space dimension $\dim(x(t)) = 100$,
- parameter-space dimension $\dim(\theta) = 100$,
- uniformly random parameters $\theta_i \leftarrow \mathcal{U}_{[0,1]}$,
- parametrization: $K_{ii} = \theta_i$.



Read Me

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