MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

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Model Reduction for Gas Networks

Nonlinear Model Order Reduction Using Empirical Gramian Matrices and Balancing

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Abstract

In modelling gas transport, or energy supply systems in general, hierarchical structures arise. Globally, a network model describes the distribution of the gas, for example by a graph; and locally, this graph's nodes and edges represent technical components such as junctions and pipes, which are of nonlinear and hyperbolic nature. Additionally, the local and global sub-models are assumed to be conservative. Hence, for large-scale networks, the simulation of the entire model's transient behaviour becomes computationally challenging. To combat this complexity, model reduction aims to obtain reduced order models which allow swift but accurate simulations. We present such a model reduction method, which is tested on a hierarchical benchmark system similar to real gas networks.

Hierarchical Model

1. **Network**
2. **Pipe**

The Network

Graph $(\mathcal{N}, \mathcal{E})$:

- Nodes: $\mathcal{N}$
- Edges: $\mathcal{E}$

Incidence Matrix (Directed Graph):

$$A \in \{-1, 0, 1\}^{|\mathcal{N}| \times |\mathcal{E}|}$$

Special Nodes:

- Supply Nodes: Pressure-based gas insertion
- Demand Nodes: Flow-based gas extraction

A Single Pipe

Transient pressure p and flow q behaviour:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{1}{s}\frac{\partial q(x,t)}{\partial x}$$
$$\frac{\partial q(x,t)}{\partial t} = -sc^2 \frac{\partial p(x,t)}{\partial t} + \frac{\lambda}{2ds}\frac{q(x,t)|q(x,t)|}{p(x,t)}$$

- Pipe model based on hyperbolic conservation laws
- 1D isothermal Euler equations:
  - Pressure is given by conservation of mass
  - Flow is given by conservation of momentum
- For details see [1]

The Gas Network Model

Spatially Discretized:

- Each edge treated as a pipe (repetitive modelling)
- Nonlinear DAE (!)

Implicit ODE Formulation:

$$|A| M_L |A^\intercal| \dot{\rho}_d = A q_+(t) - d(t) - |A| M_L |A_S^\intercal| \dot{s}(t)$$
$$\dot{q}_+ = M_a A^\intercal \rho_d - \frac{\lambda}{D} \frac{q_+|q_+|}{|A_S^\intercal| s(t) + |A^\intercal| \rho_d} + M_a A_S^\intercal s(t)$$

- Variables: densities ρ_d, mass inflows q_+ (per pipe)
- Network: supply incidence A_S, supply $s(t)$, demand $d(t)$
- Pipe specifications: M_L, M_a, λ, D
- For details see [2]

Input-Output System

Input:

$$u(t) := \begin{pmatrix} u_p(t) \\ u_q(t) \end{pmatrix} \begin{matrix} \leftarrow \text{Supply Pressure} \\ \leftarrow \text{Demand Mass-Flow} \end{matrix}$$

Output:

$$y(t) := \begin{pmatrix} y_q(t) \\ y_p(t) \end{pmatrix} \begin{matrix} \leftarrow \text{Supply Mass-Flow} \\ \leftarrow \text{Demand Pressure} \end{matrix}$$

$\Rightarrow$ **Control-Affine Descriptor Input-Output System:**

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + h(x(t)), \\ y(t) &= Cx(t), \\ x(0) &= x_0 \end{aligned}$$

Model Reduction

General Nonlinear **Full Order Model** (FOM):

$$\begin{aligned} 0 &= f(\dot{x}(t), x(t), u(t)), \\ y(t) &= g(x(t), u(t)), \\ x(0) &= x_0 \end{aligned}$$

Reduced Order Model (ROM):

$$\begin{aligned} 0 &= f_r(\dot{x}_r(t), x_r(t), u(t)), \\ y_r(t) &= g_r(x_r(t), u(t)), \\ x_r(0) &= x_{r,0}, \end{aligned}$$

such that:

$$\dim(x_r(t)) =: n \ll N := \dim(x(t)),$$
$$\|y - y_r\| \ll 1$$

Projection-Based ROM:

$$\begin{aligned} 0 &= Vf(U\dot{x}_r(t), Ux_r(t), u(t)), \\ y_r &= g(Ux_r(t), u(t)), \\ x_r(0) &= Vx_0, \end{aligned}$$

with:

$$U \in \mathbb{R}^{N \times n}, \quad V \in \mathbb{R}^{n \times N}, \quad VU = \mathbb{1}$$

Empirical Gramians

Balancing Idea:

- Focus on input-output coherence
- Consider:
  - Input-to-State behaviour (Controllability)
  - State-to-Output behavior (Observability)
- Balance Controllability and Observability
  - Balanced Truncation (Petrov-Galerkin Projection)
  - Direct Truncation (Galerkin Projection)

System Gramians:

- Linear systems
  - Controllability Gramian Matrix W_C
  - Observability Gramian Matrix W_O
  - Cross Gramian Matrix W_X
- Extract truncated projections from Gramians

Empirical Gramians:

- Applicable to nonlinear systems
- Data-driven computation
- For linear systems equal to system Gramians
- Simple example ($E = 1$):

$$W_X = \int_0^\infty e^{At} BC e^{At} \, dt \overset{\text{SVD}}{=} UDV$$

Numerical Results

Network:

- 32 Nodes (1 Supply Node, 1 Demand Node)
- 31 Edges (Long Pipeline Model)

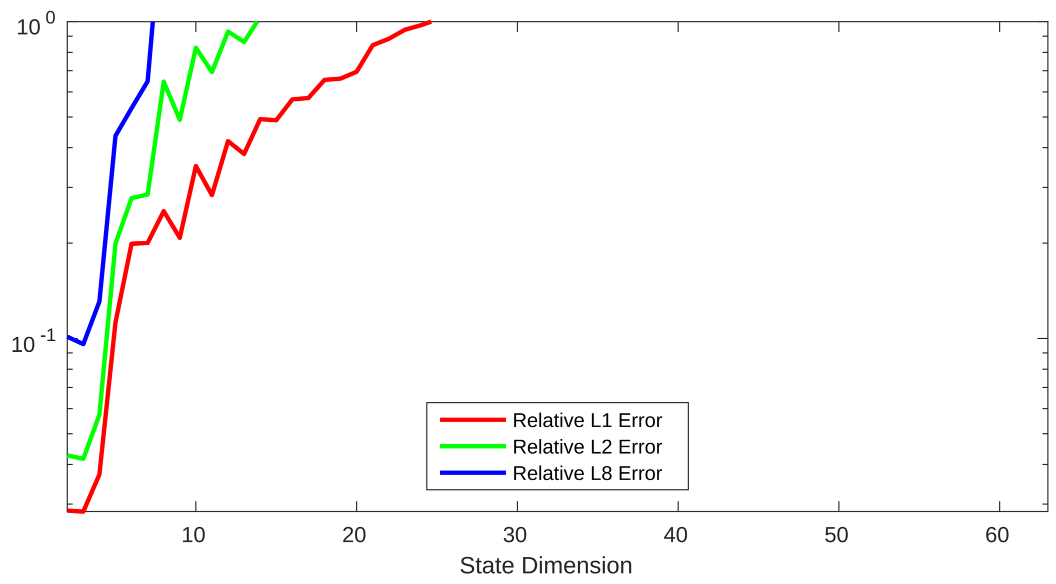
Empirical-Gramian-Based Model Reduction:

1. Global Projection [4]
2. Pressure and Mass-Flow Component Projections [5]

Computation of Empirical Gramians: see [6]

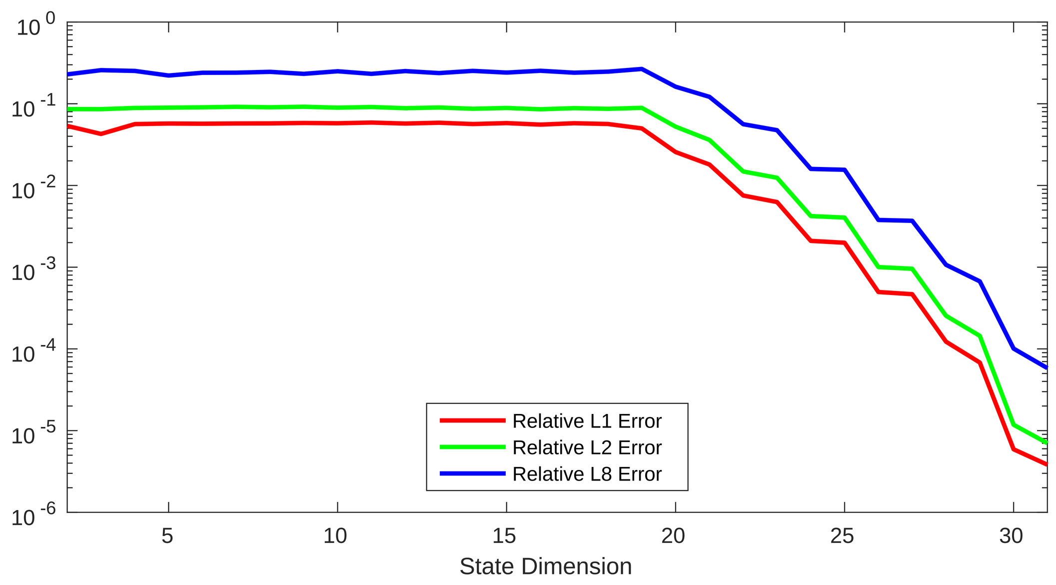
Model Reduction Errors

Method 1 (global projection):



→ Large model reduction errors for low orders.
→ Unstable reduced order models for increasing orders.

Method 2 (component projections):



→ Accelerated reduced order model evaluation.
→ Slowly decreasing model reduction error.

Outlook

This is an initial test. Upcoming:

- Improved empirical-Gramian-based model reduction
- Large-scale networks
- Additional gas network elements (i.e.: Compressors)
- Model enhancements (i.e.: Compressibility, Incline)
- Uncertainties
- Parametrizations

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