Model Reduction for Dynamic Connectivity Models

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About

Dynamic Connectivity Models
In the neurosciences dynamic connectivity models describe the mesoscale information propagation between brain regions in terms of associated functional measurements. A network submodel models the average neuronal activity, which a forward submodel transforms to the observable measurements.

Model Order Reduction
Model order reduction refers to the process of computing low-dimensional surrogate models exhibiting the same dynamics as the original model. In this setting, gramian-based combined parametric state-space reduction and parameter-space reduction for nonlinear input-output systems is considered [1].

Dynamic Causal Modelling
Dynamic causal modelling [2] is a framework for hypothesis testing based on bayesian inference constrained by dynamic connectivity models, which are given by nonlinear input-output systems. The network submodel's connectivity is parametrized and inferred from functional measurements.

← 1. Record Data

Start here!

- Stimulus Experiments

$$u(t) \overset{?}{\to} y(t)$$

- Functional neuroimaging:
  - EEG & MEG
  - fMRI & fNIRS

Correlation: ✗
Causality: ✓

2. Setup Model →

- Data model with Gaussian noise:

$$y_d(t) = y(t;\theta) + \varepsilon$$

- Network and Forward model:

$$\dot{x}(t) = f(x(t), u(t), \theta)$$
$$y(t) = g(x(t), u(t), \theta)$$

State Reduction

- Large state-space, small input- and output-spaces:

$$N := \dim(x(t)) \gg 1$$
$$M := \dim(u(t)) \ll N, Q := \dim(y(t)) \ll N$$

- Truncated-projection-based model reduction:

$$x_r(t) = Vx(t) \to x(t) \approx Ux_r(t)$$

- Target reduced order model property:

$$\|y(\theta) - y_r(\theta)\| \ll 1$$

- Empirical-Gramian-based model order reduction:

$$x^{m=1\dots M}(t), \quad y^{n=1\dots N}(t)$$

- Empirical cross Gramian [4]:

$$W_X = \sum_{m=1}^{M} \int \Psi^m(t)\mathrm{d}t \in \mathbb{R}^{N\times N}, \Psi_{ij}^m = \langle x_i^m(t), y_m^j(t)\rangle$$

- Non-symmetric (average) cross Gramian:

$$W_Z = \sum_{i=1}^{M}\sum_{j=1}^{Q} W_{X,ij}$$

- Direct Truncation:

$$W_X \overset{\text{TSVD}}{=} UDV \to V := U^\intercal$$

←2a. Combined Reduction→

(Combined State and Parameter Reduction)

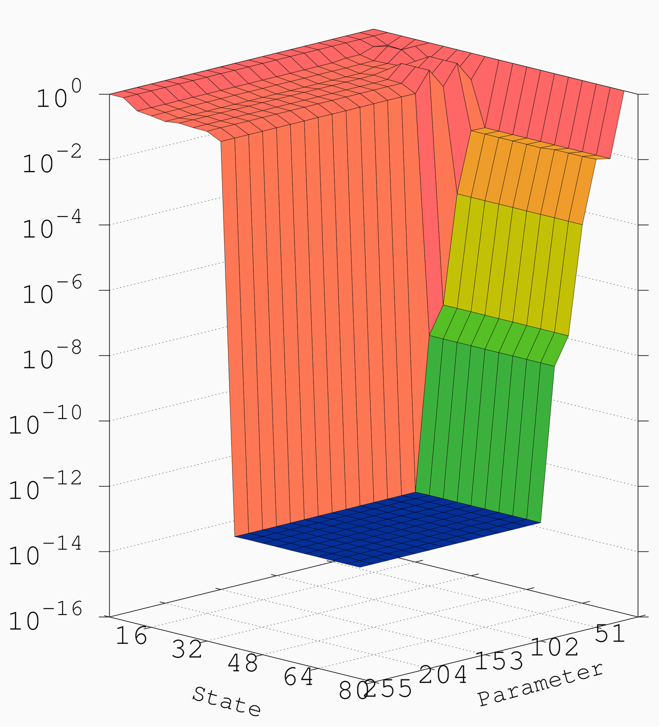

Fig. 1: Relative L_2 combined reduction (output) error for varying reduced state and parameter dimension of the fMRI & fNIRS dynamic causal model from [2].

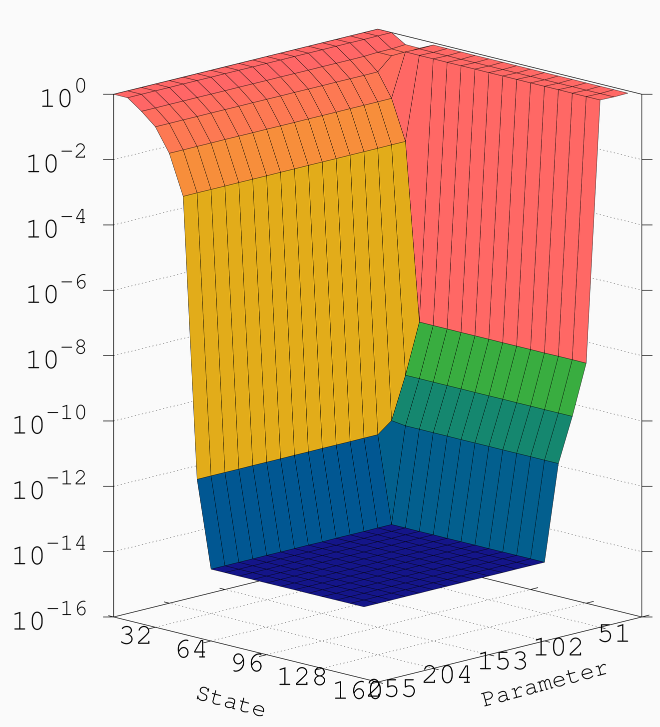

Fig. 2: Relative L_2 combined reduction (output) error for varying reduced state and parameter dimension of the EEG & MEG dynamic causal model from [3].

Parameter Reduction

- High dimensional parameter-space:

$$\dim(\theta) \gg 1$$

- Truncated-projection-based model reduction:

$$\theta_r(t) := \Lambda\theta \to \theta \approx \Pi\theta_r$$

- Target reduced order model property:

$$\|y(\theta) - y_r(\theta_r)\| \ll 1$$

- Augmented System:

$$\begin{pmatrix}\dot{x} \\ \dot{\theta}\end{pmatrix} = \begin{pmatrix} f(x(t), u(t), \theta) \\ 0 \end{pmatrix}$$

- Empirical cross-identifiability Gramian [4]:

$$W_J = \begin{pmatrix} W_X & W_m \\ 0 & 0 \end{pmatrix} \to W_{\tilde{I}} = W_m^\intercal (W_X + W_X^\intercal)^{-1} W_m$$

- Direct Truncation:

$$W_{\tilde{I}} \overset{\text{TSVD}}{=} \Pi\Delta\Lambda \to \Lambda := \Pi^\intercal$$

- Reduced Order Model:

$$\dot{x}_r(t) = V f(Ux_r(t), u(t), \Pi\theta_r)$$
$$y_r(t) = g(Ux_r(t), u(t), \Pi\theta_r)$$

← 3. Inverse Problem

- Bayesian inference:

$$P(\theta|y) \propto P(y|\theta)P(\theta)$$

- Optimization of posterior:

$$P(\theta|y) \propto \arg\min_\theta \|y_d - y(\theta)\|_{L_2}^2 + \beta\|\theta\|_2^2$$

4. Evaluation →

- Maximum-A-Posteriori Estimate:

$$\text{vec}^{-1}(P(\theta|y))$$

- Model comparison based on model evidence

$$P(y(\theta_1)) \overset{?}{>} P(y(\theta_2))$$

Summary & Conclusion

- Respective sources for cross-gramian-based information extraction:
  - State-space reduction: input-to-output coherence
  - Parameter-space reduction: state-to-output coherence
- The empirical-cross-gramian-based combined state and parameter reduction applies to any model of the form:

$$\dot{x}(t) = f(x(t), u(t), \theta), \quad y(t) = g(x(t), u(t), \theta)$$

- For the dynamic causal models, low-dimensional spaces contain the principal information.

What's Next?

- Instead of specific models for each neuroimaging technique:
- A universal connectivity model;
- promising candidate: Hyperbolic Network Model [5]:

$$\dot{x}(t) = A(\theta)\tanh(Kx(t)) + Bu(t), \quad y(t) = Cx(t)$$

- Preliminary results show:
  - Works well for EEG & MEG due to the similar nonlinearities
  - Has to be tuned for fMRI & fNIRS.

README

[1] C. Himpe. **Combined State and Parameter Reduction for Nonlinear Systems with an Application in Neuroscience**. PhD Thesis, University of Münster, 2016.

[2] K.J. Friston, L.M. Harrison, and W. Penny. **Dynamic causal modelling**. *NeuroImage*, 19(4):1273–1302, 2003.

[3] R.J. Moran, S.J. Kiebel, K.E. Stephan, R.B. Reilly, J. Daunizeau, and K.J. Friston. **A neural mass model of spectral responses in electrophysiology**. *NeuroImage*, 37(3):706–720, 2007.

[4] C. Himpe and M. Ohlberger. **Cross-Gramian Based Combined State and Parameter Reduction for Large-Scale Control Systems**. *Mathematical Problems in Engineering*, 2014:1–13, 2014.

[5] Y. Quan, H. Zhang, and L. Cai. **Modeling and Control Based on a New Neural Network Model**. *In Proceedings of the American Control Conference*, volume 3, pages 1928–1929, 2001.

RUNME

The presented numerical results are part of [1] and can be reproduced using the companion code: