MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

20 YEARS 1998–2018

Math Energy

Federal Ministry for Economic Affairs and Energy

Complexity Reduction in Gas Networks via Model Reduction

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Abstract

The growing infeed of renewable energy requires a change in management of gas networks, as supply and demand become increasingly volatile. To ensure safe operation of the gas network, many scenario simulations of a large-scale model are conducted prior to the dispatch. Model reduction alleviates the associated computational complexity by providing surrogate models with resemblant behavior.

Gas Network Model

Spatially Discrete Index-Reduced Isothermal Euler Equations:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & A_{pq} \\ A_{qp} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 & B_p \\ B_q & 0 \end{pmatrix} \begin{pmatrix} u_s \\ u_d \end{pmatrix} + \begin{pmatrix} 0 \\ f_q(p, q, \theta) \end{pmatrix}$$

$$\begin{pmatrix} y_d \\ y_s \end{pmatrix} = \begin{pmatrix} C_p & 0 \\ 0 & C_q \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

- Pressure: $p : \mathbb{R} \to \mathbb{R}^{N_p}$
- Mass-Flux: $q : \mathbb{R} \to \mathbb{R}^{N_q}$
- Supply Pressure (Input): $u_s : \mathbb{R} \to \mathbb{R}^{M_s}$
- Demand Mass-Flux (Input): $u_d : \mathbb{R} \to \mathbb{R}^{M_d}$
- Supply Mass-Flux (Output): $y_d : \mathbb{R} \to \mathbb{R}^{Q_d}$
- Demand Pressure (Output): $y_s : \mathbb{R} \to \mathbb{R}^{Q_d}$

A Step-By-Step Guide

Outline:

1. Generic Model Reduction
2. Linear Model Reduction
3. Affine Model Reduction
4. Structured Model Reduction
5. Parametric Model Reduction
6. Combined Reduction
7. Hyper Reduction

1a. Generic Model

(Possibly Nonlinear) Input-Output System:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

- Input: $u : \mathbb{R} \to \mathbb{R}^M$
- State: $x : \mathbb{R} \to \mathbb{R}^N$
- Output: $y : \mathbb{R} \to \mathbb{R}^Q$
- Vectorfield: $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$
- Output Functional: $g : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^Q$

1b. Generic Model Reduction

Reduced Order Model:

$$\begin{aligned} \dot{x}_r(t) &= f_r(x_r(t), u(t)) \\ \tilde{y}(t) &= g_r(x_r(t), u(t)) \end{aligned}$$

- Reduced State: $x_r : \mathbb{R} \to \mathbb{R}^n$
- Reduced State-Space Dimension: $n \ll N$
- Approximate Output: $\tilde{y} : \mathbb{R} \to \mathbb{R}^Q$
- Reduced Vectorfield: $f_r : \mathbb{R}^n \times \mathbb{R}^M \to \mathbb{R}^n$
- Reduced Output Functional: $g_r : \mathbb{R}^n \times \mathbb{R}^M \to \mathbb{R}^Q$

1c. Projection-Based Model Reduction

(Low-Dimensional) Projected Input-Output System:

$$\begin{aligned} \dot{x}_r(t) &= V^\intercal f(Ux_r(t), u(t)) \\ \tilde{y}(t) &= \quad g(Ux_r(t), u(t)) \end{aligned}$$

- Reducing Truncated Projection: $U \in \mathbb{R}^{N \times n}$
- Reconstructing Truncated Projection: $V \in \mathbb{R}^{N \times n}$
- Bi-Orthogonality: $V^\intercal U = \mathbb{1}$
- Reduced State: $x_r(t) := V^\intercal x(t)$
- Model Reduction Error: $\|y - \tilde{y}\| \ll 1$

2. Linear Model Reduction

Reduced Linear Model:

$$\begin{aligned} \dot{x}_r(t) &= (V^\intercal AU)x_r(t) + (V^\intercal B)u(t) \\ \tilde{y}(t) &= (CU)x_r(t) \end{aligned}$$

- Projections can be applied a-priori.
- Reduced System Matrix $A_r := V^\intercal AU \in \mathbb{R}^{n \times n}$
- Reduced Input Matrix: $B_r := V^\intercal B \in \mathbb{R}^{n \times M}$
- Reduced Output Matrix: $C_r := CU \in \mathbb{R}^{Q \times n}$
- Extensive theory exists for linear (linearized) models.

3. Affine Model Reduction

Affinely Reduced Input-Output System:

$$\begin{aligned} \dot{x}_r(t) &= V^\intercal f(\bar{x} + Ux_r(t), u(t)) \\ \tilde{y}(t) &= \quad g(\bar{x} + Ux_r(t), u(t)) \end{aligned}$$

- Steady-State: $\bar{x} \in \mathbb{R}^N$
- Reduced State: $x_r(t) := V^\intercal x(t) - \bar{x}$
- Reconstructed State: $x(t) \approx Ux_r(t) + \bar{x}$
- Simple "nonlinear" model reduction method.
- Useful for nonlinear systems.

4. Structured Model Reduction

Structured Reduced Order Model:

$$\begin{pmatrix} \dot{p}_r(t) \\ \dot{q}_r(t) \end{pmatrix} = \begin{pmatrix} V_p^\intercal f_p(U_p p_r(t), U_q q_r(t), u(t)) \\ V_q^\intercal f_q(U_p p_r(t), U_q q_r(t), u(t)) \end{pmatrix}$$
$$\tilde{y}(t) = g(U_p p_r(t), U_q q_r(t), u(t))$$

- Reducing Projections: $U_p \in \mathbb{R}^{N_p \times n_p}$, $U_q \in \mathbb{R}^{N_q \times n_q}$
- Reconstructing Projections: $V_p \in \mathbb{R}^{N_p \times n_p}$, $V_q \in \mathbb{R}^{N_q \times n_q}$
- Bi-Orthogonality: $V_p^\intercal U_p = \mathbb{1}$, $V_q^\intercal U_q = \mathbb{1}$
- Reduced States: $p_r(t) := V_p^\intercal p(t)$, $q_r(t) := V_q^\intercal q(t)$

5. Parametric Model Reduction

Parametric Input Output System:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), \theta) \\ y(t) &= g(x(t), u(t), \theta) \end{aligned}$$

- Parameter: $\theta \in \mathbb{R}^P$
- Vectorfield: $f : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^N$
- Output Functional: $g : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$
- Goal: Find projections U, V valid over $\Theta \subset \mathbb{R}^P$
- Parametric Approximate Output: $\|y(\theta) - \tilde{y}(\theta)\| \ll 1$

6. Combined Reduction*

Combined State and Parameter Reduction:

$$\begin{aligned} \dot{x}_r(t) &= V^\intercal f(Ux_r(t), u(t), \Pi\theta_r) \\ \tilde{y}(t) &= \quad g(Ux_r(t), u(t), \Pi\theta_r) \end{aligned}$$

- Reducing Truncated Projection: $\Pi \in \mathbb{R}^{P \times p}$
- Reconstructing Truncated Projection: $\Lambda \in \mathbb{R}^{P \times p}$
- Bi-Orthogonality: $\Lambda^\intercal U = \mathbb{1}$
- Reduced Parameter: $\theta_r := \Lambda^\intercal \theta \in \mathbb{R}^{p \ll P}$
- Model Reduction Error: $\|y(\theta) - \tilde{y}(\theta_r)\| \ll 1$

7. Hyper Reduction*

Lifting Bottleneck:

$$\begin{aligned} \dot{x}_r(t) &= V^\intercal f(Ux_r(t), u(t)) \\ \tilde{y}(t) &= \quad g(Ux_r(t), u(t)) \end{aligned}$$

- f_r requires evaluation of f.
- x_r needs lifting to x.
- Approximate f_r by interpolating between data points of f.
- (Likely) Not necessary for gas networks.
- Algorithm: (Discrete) Empirical Interpolation Method

Reduced Gas Network Model

Putting it all together:

$$\begin{pmatrix} \dot{p}_r \\ \dot{q}_r \end{pmatrix} = A_r \begin{pmatrix} p_r \\ q_r \end{pmatrix} + \begin{pmatrix} \bar{p}_r \\ \bar{q}_r \end{pmatrix} + B_r \begin{pmatrix} u_s \\ u_d \end{pmatrix} + \begin{pmatrix} 0 \\ f_{q,r}(p_r, q_r, \theta) \end{pmatrix}$$

$$\begin{pmatrix} \tilde{y}_d \\ \tilde{y}_s \end{pmatrix} = C_r \begin{pmatrix} p_r \\ q_r \end{pmatrix} + \begin{pmatrix} \bar{y}_d \\ \bar{y}_s \end{pmatrix}$$

- Reduced Pressure: $p_r(t) := V_p^\intercal p(t) - \bar{p}$
- Reduced Mass-Flux: $q_r(t) := V_q^\intercal q(t) - \bar{q}$
- $f_{q,r}(p_r, q_r, \theta) := V_q^\intercal f_q(\bar{p} + U_p p_r, \bar{q} + U_q q_r, \theta)$

Projection Computation

Nonlinear Data-Driven Methods Considered:

- Empirical Balanced Truncation
- Empirical Cross Gramian
- Empirical Non-Symmetric Cross Gramian
- Proper Orthogonal Decomposition
- Dynamic Mode Decomposition

Furthermore ...

Model reduction methods have different properties, i.e.:

- Galerkin ($V = U$) or Petrov-Galerkin ($V \neq U$) projections
- Target Error Norm: L_1, L_2, L_∞, H_2, H_∞, ...
- Stability Preservation
- (Sharp) Error Indicators
- Input-Output Coherence

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README

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- **MORwiki – The Model Order Reduction Wiki.** `http://modelreduction.org`

Acknowledgement: Supported by the German Federal Ministry for Economic Affairs and Energy, in the joint project: "**MathEnergy** – Mathematical Key Technologies for Evolving Energy Grids", sub-project: Model Order Reduction (Grant number: 0324019**B**).