MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

Math Energy

Federal Ministry for Economic Affairs and Energy

Solving Matrix Equations via Empirical Gramians

Christian Himpe (iD 0000-0003-2194-6754)[1]

[1]Max Planck Institute for Dynamics of Complex Technical Systems Magdeburg, Computational Methods in Systems and Control Theory

About

In system theory, the so-called system Gramian matrices are operators encoding certain properties of an underlying input-output system. Usually, these system Gramians are computed as solutions to matrix equations, such as the Lyapunov equation and Sylvester equation. This means, the solution to certain matrix equations coincides with these system Gramians. Now, if the system Gramians are computable by other means than matrix equations, they still represent solutions to matrix equations. Empirical Gramians are such an alternative for system Gramian computation, which are based on their system-theoretic operator definition, and practically obtained via quadrature. This contribution explores the connection between matrix equations, system Gramians and empirical Gramians, and proposes empirical Gramians as potential solver for matrix equations.

Continuous-Time Linear Time-Invariant System	Discrete-Time Linear Time-Invariant System
$\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t)$	$x_{k+1} = Ax_k + Bu_k$ $y_k = Cx_k$

Lyapunov Equation

Lyapunov Equation		Factor RHS		Controllability Gramian		Empirical Controllability Gramian		Low Rank Empirical Gramian
$AW + WA^\intercal = -Y$	$\overset{\text{rrSVD}}{\to}$	$AW + WA^\intercal = -BB^\intercal$	$\overset{\text{IbP}}{\Leftrightarrow}$	$W = \int_0^\infty e^{At}BB^\intercal e^{A^\intercal t}dt$	$\overset{\int}{\approx}$	$\widetilde{W} = \frac{\Delta t}{M}\sum_{m=1}^{M} X_m(A,B)X_m(A,B)^\intercal$	$\overset{\text{tSVD}}{\approx}$	$\widetilde{W} = \frac{\Delta t}{M}\sum_{m=1}^{M} \widetilde{X}_m(A,B)\widetilde{X}_m(A,B)^\intercal$

Sylvester Equation

Sylvester Equation		Factor RHS		Cross Gramian		Empirical Cross Gramian		Low Rank Empirical Gramian
$AW + WF = -Y$	$\overset{\text{rrSVD}}{\to}$	$AW + WF = -BH$	$\overset{\text{IbP}}{\Leftrightarrow}$	$W = \int_0^\infty e^{At}BHe^{Ft}dt$	$\overset{\int}{\approx}$	$\widetilde{W} = \frac{\Delta t}{M}\sum_{m=1}^{M} X_m(A,B)X_m(F,H)^\intercal$	$\overset{\text{tSVD}}{\approx}$	$\widetilde{W} = \frac{\Delta t}{M}\sum_{m=1}^{M} \widetilde{X}_m(A,B)\widetilde{X}_m(F,H)^\intercal$

(Symmetric) Stein Equation

Stein Equation		Factor RHS		Controllability Gramian		Empirical Controllability Gramian		Low Rank Empirical Gramian
$AWA^\intercal - W = -Y$	$\overset{\text{rrSVD}}{\to}$	$AWA^\intercal - W = -BB^\intercal$	$\Leftrightarrow$	$W = \sum_{k=0}^{\infty} A^k BB^\intercal (A^\intercal)^k$	$\overset{\int}{\approx}$	$\widetilde{W} = \frac{1}{M}\sum_{m=1}^{M} X_m(A,B)X_m(A,B)^\intercal$	$\overset{\text{tSVD}}{\approx}$	$\widetilde{W} = \frac{1}{M}\sum_{m=1}^{M} \widetilde{X}_m(A,B)\widetilde{X}_m(A,B)^\intercal$

Riccati Equation

Riccati Equation		Factor RHS		Newton Iteration via Lyapunov Equations
$AW + WA^\intercal = -Y - WZW$	$\overset{\text{rrSVD}}{\to}$	$AW + WA^\intercal = -BB^\intercal - WC^\intercal CW$	$\to$	$(A - C^\intercal CW_k)W_{k+1} + W_{k+1}(A - C^\intercal CW_k)^\intercal = -(BB^\intercal + W_k C^\intercal CW_k),\ W_0 = 0$

More Matrix Equations

- Generalized Lyapunov Equation: $AWE^\intercal + EWA^\intercal = BB^\intercal$
- Generalized Sylvester Equation: $AWE + EWA = BC$
- Generalized Stein Equation: $A^T WA + EWE^\intercal = BB^\intercal$
- Cross-Riccati Equation: $AW + WA = -BC - WBCW$

Generalized Continuous-Time Linear Time-Invariant System	Generalized Discrete-Time Linear Time-Invariant System
$E\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t)$	$Ex_{k+1} = Ax_k + Bu_k$ $y_k = Cx_k$

Notes

- Choice of time-stepping solver is crucial.
- Right-hand-sides need to be low-rank.
- Time varying systems relate to matrix differential equation:
  $\dot{x}(t) = A(t)x(t) + B(t)u(t) \leftrightarrow \dot{X}(t) = A(t)X(t) + X(t)A(t)^\intercal + B(t)B(t)^\intercal$

README

- L. Fortuna, A. Gallo, C. Guglielmino, G. Nunnari. **On the solution of a nonlinear matrix equation for MIMO symmetric realizations**, System and Control Letters, 11(1): 79–82, 1988. 10.1016/0167-6911(88)90115-6
- S. Lall, J.E. Marsden, S. Glavaski. **Empirical Model Reduction of Controlled Nonlinear Systems**. Proceedings of the IFAC World Congress, F: 473–478, 1999. doi:10.1016/S1474-6670(17)56442-3
- P. Benner. **Efficient Algorithms for Large-Scale Quadratic Matrix Equations**. Proceeding in Applied Mathematics and Mechanics, 1(1): 492–495, 2002. 10.1002/1617-7061(200203)1:1<492::AID-PAMM492>3.0.CO;2-W
- C.W. Rowley. **Model Reduction for Fluids, Using Balanced Proper Orthogonal Decomposition**. International Journal of Bifurcation and Chaos, 15(3): 997–1013, 2005. doi:10.1142/S0218127405012429
- J.R. Singler. **Approximate Low Rank Solutions of Lyapunov Equations via Proper Orthogonal Decomposition**. Proceedings of the American Control Conference: 267–272, 2008. doi:10.1109/ACC.2008.4586502
- B. Kramer. **Solving Algebraic Riccati Equations via Proper Orthogonal Decomposition**. Proceedings of the 19th IFAC World Congress: 7767–7772, 2014. 10.3182/20140824-6-ZA-1003.02477
- C. Himpe. **emgr — The Empirical Gramian Framework**. Algorithms 11(7): 91, 2018. doi:10.3390/a11070091

Acknowledgement: Supported by the German Federal Ministry for Economic Affairs and Energy, in the joint project: “**MathEnergy** – Mathematical Key Technologies for Evolving Energy Grids”, sub-project: Model Order Reduction (Grant number: 0324019**B**).