MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

ICERM

Math Energy

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emgr – EMpirical GRamian Framework

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Introduction

Abstract:
A wide range of industrial applications from control engineering, systems engineering, computational engineering or uncertainty quantification are addressed by mathematical system theory, including reduced order modeling. For linear systems, an essential tool for these tasks are system Gramian matrices. Yet, linear or linearized systems may not suffice to model and simulate dynamics of complex technical systems. So, for nonlinear or parametric systems, the data-driven empirical system Gramian matrices generalize linear methods, and the empirical Gramian framework - `emgr` - is an open-source software toolbox for their computation.

Applications:

- Model Order Reduction
- Decentralized Control
- Optimal Placement
- Sensitivity Analysis
- Structural Identifiability
- Parameter Reduction
- Combined State and Parameter Reduction
- Nonlinearity Quantification
- Uncertainty Quantification
- System Characterization

Empirical System Gramians

Given a nonlinear (and parametric) input-output system, with vector field $\dot{x}(t) = f(t, x(t), u(t), \theta)$ and output functional $y(t) = g(t, x(t), u(t), \theta)$:

Reachability – Input-To-State

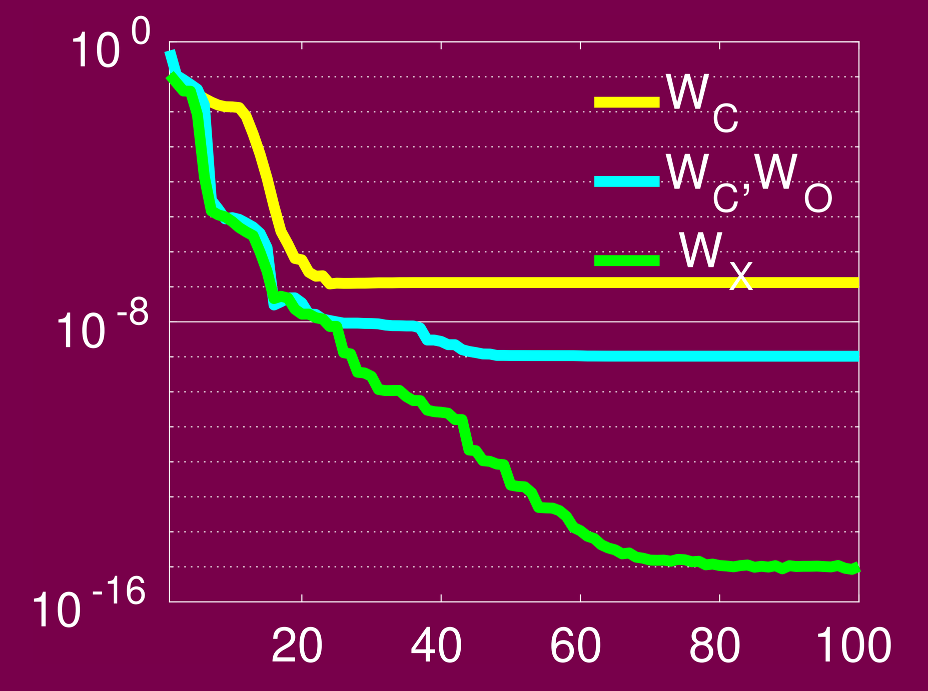
Perturb m-th component of training input u,

compute associated state trajectory x^{km},

and form average Gramian matrix:

$$W_R := \frac{1}{|S_u|} \sum_{k=1}^{|S_u|} \sum_{m=1}^{M} c_k^{-2} \int_0^\infty x^{km}(t) \otimes x^{km}(t)\,\mathrm{d}t,$$

$\to$ Empirical Reachability Gramian



Reducibility of nonlinear RC cascade by dominant subspaces.

Observability – State-To-Output

Perturb n-th component of training initial state x_0,

compute associated output trajectory $y^{\ell n}$,

and form average Gramian matrix:

$$W_O := \frac{1}{|S_x|} \sum_{\ell=1}^{|S_x|} d_\ell^{-2} \int_0^\infty \Psi^\ell(t)\,\mathrm{d}t, \quad \Psi_{ij}^\ell = \langle y^{\ell i}, y^{\ell j} \rangle$$

$\to$ Empirical Observability Gramian

Minimality – Input-To-Output

Perturb m-th component of training input u and compute associated state trajectory x^{km},

perturb n-th component of training initial state x_0 and compute output trajectory $y^{\ell n}$,

and form average cross-correlation matrix:

$$W_X := \frac{1}{|S_u||S_x|} \sum_{k=1}^{|S_u|} \sum_{\ell=1}^{|S_x|} \sum_{m=1}^{M} c_k^{-2} d_\ell^{-2} \int_0^\infty \Psi^{k\ell m}(t)\,\mathrm{d}t, \quad \Psi_{ij}^{k\ell m}(t) = x_i^{km}(t) \cdot y_m^{\ell j}(t)$$

$\to$ Empirical Cross Gramian

Sensitivity – Parameter Reachability

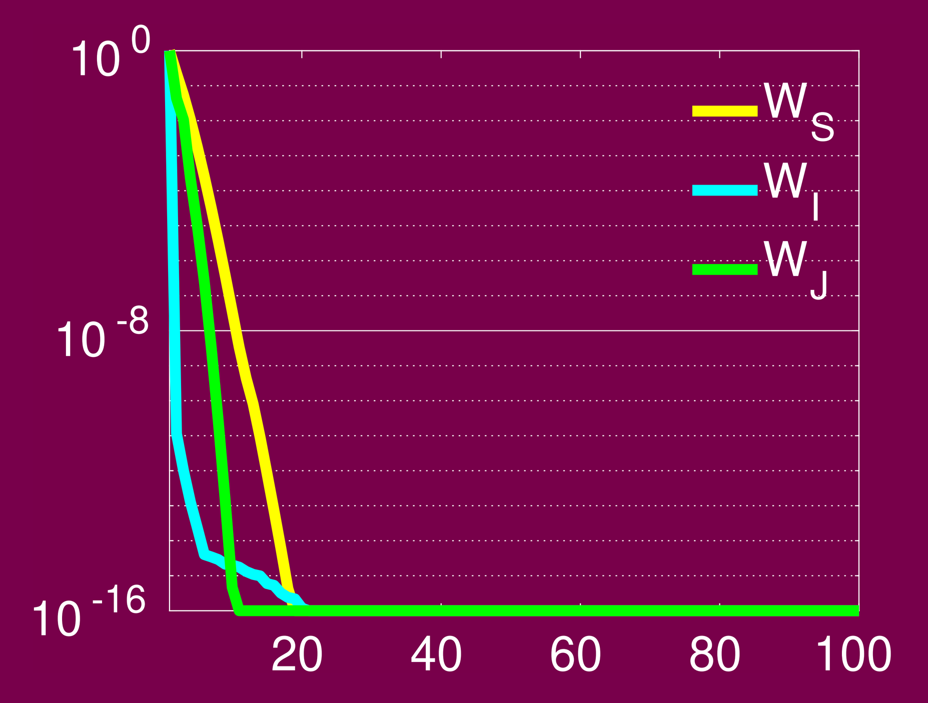
Treat i-th component of parameter θ as extra input,

compute empirical reachability Gramian $W_R(\theta_i)$,

and form diagonal matrix from traces:

$$W_{S,ii} := \operatorname{tr}\big(W_R(\theta_i)\big)$$

$\to$ Empirical Sensitivity Gramian



Identifiability of nonlinear RC cascade resistors.

Identifiability – Parameter Observability

Treat parameter θ as (constant) extra states, compute

augmented empirical observability Gramian $\begin{pmatrix} W_O & W_M \\ W_M^\intercal & W_\theta \end{pmatrix}$,

and form Schur complement:

$$W_I := W_\theta - W_M^\intercal W_O^{-1} W_M$$

$\to$ Empirical Identifiability Gramian

Combined State Minimality and Parameter Observability

Treat parameter θ as (constant) extra states,

compute augmented empirical cross Gramian $\begin{pmatrix} W_X & W_M \end{pmatrix}$,

and form Schur complement of symmetric part:

$$W_{\bar{I}} := -\frac{1}{4} W_M^\intercal (W_X + W_X^\intercal)^{-1} W_M$$

$\to$ Empirical Cross-Identifiability Gramian

README

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