HAPOD – Hierarchical Approximate POD

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Abstract

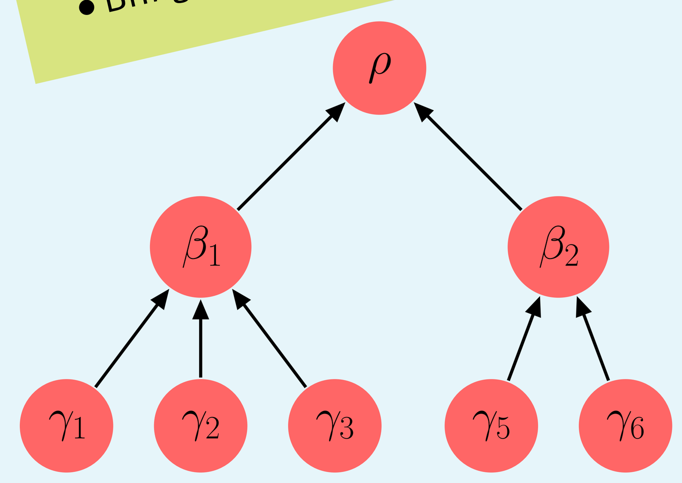
Proper orthogonal decomposition (POD) is a widely-used model order reduction technique for the computation of surrogate reduced state spaces from given solution snapshot data. However, performing a POD is often a computationally demanding task since the complexity relates quadratically to the number of snapshots.
A nearby solution to overcome this limitation is to compute, when and where available, PODs of subsets of the global snapshot set, and then to use the resulting POD modes as input for an additional POD. We formalise this approach as "hierarchical approximate POD" (HAPOD), allowing arbitrary trees of localized PODs, making HAPOD suitable for distributed, heterogeneous compute environments.
As special cases of the HAPOD we consider a "distributed approximate POD" (DAPOD) and a "rolling approximate POD" (RAPOD), for which we present numerical results for a model reduction benchmark problem.

HAPOD – Hierarchical Approximate POD

HAPOD can be easily implemented on top of any existing POD code. Given a set $\mathcal{S}$ of snapshot vectors, we assume that $\mathrm{POD}(\mathcal{S}, \varepsilon)$ computes the first N POD mode / singular value pairs such that:

- $\mathcal{O}(|\mathcal{S}|^2) \to \mathcal{O}(|\mathcal{S}|\log(|\mathcal{S}|))$!
- Cloud-friedly!
- Bring-your-own-SVD!

$$N = |\,\mathrm{POD}(\mathcal{S}, \varepsilon)| = \min_{N' \in \mathbb{N}} \left\{ \left(\sum_{n=N'+1}^{\infty} \sigma_n^2(\mathcal{S}) \right)^{1/2} \leq \sqrt{|S|} \cdot \varepsilon \right\} = \min_{N' \in \mathbb{N}} \left(\frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \| s - P_{\mathrm{span}\{\text{first } N' \text{ POD modes}\}}(s) \|^2 \right)^{1/2} \leq \varepsilon$$



HAPOD Input:

γ_i: snapshot vectors $\mathcal{S}_{\gamma_i}$
β_i: outputs of child nodes
ρ: outputs of child nodes

HAPOD Output:

γ_i: $\mathrm{POD}(\text{input}, \varepsilon(\gamma_i))$ scaled by singular values (pass input if $\varepsilon(\gamma_i) = 0$)
β_i: $\mathrm{POD}(\text{input}, \varepsilon(\beta_i))$ scaled by singular values
ρ: $\mathrm{POD}(\text{input}, \varepsilon(\rho))$

Abstract Error Bound:

$$\left(\frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \| s - P(s) \|^2 \right)^{1/2} \leq \sum_{l=1}^{L} \max_{L(\alpha)=l} \frac{\sqrt{M_\alpha}}{\sqrt{|\mathcal{S}_\alpha|}} \cdot \varepsilon(\alpha)$$

Abstract Mode Bound:

$$|\,\mathrm{HAPOD}(\mathcal{S}, \rho, \varepsilon)| \leq \left| \mathrm{POD}\left(\mathcal{S}, \frac{\sqrt{M_\rho}}{\sqrt{|\mathcal{S}|}} \cdot \varepsilon(\rho) - \sum_{l=1}^{L-1} \max_{L(\alpha)=l} \frac{\sqrt{M_\alpha}}{\sqrt{|\mathcal{S}_\alpha|}} \cdot \varepsilon(\alpha) \right) \right|$$

Choice of Error Tolerances:

$$\varepsilon(\rho) := \frac{\sqrt{|S|}}{\sqrt{M_\rho}} \cdot (1 - \omega) \cdot \varepsilon^*, \qquad \varepsilon(\alpha) := \frac{\sqrt{|S_\alpha|}}{\sqrt{M_\alpha (L-1)}} \cdot \omega \cdot \varepsilon^*$$

Error Bound:

$$\left(\frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \| s - P(s) \|^2 \right)^{1/2} \leq \varepsilon^*$$

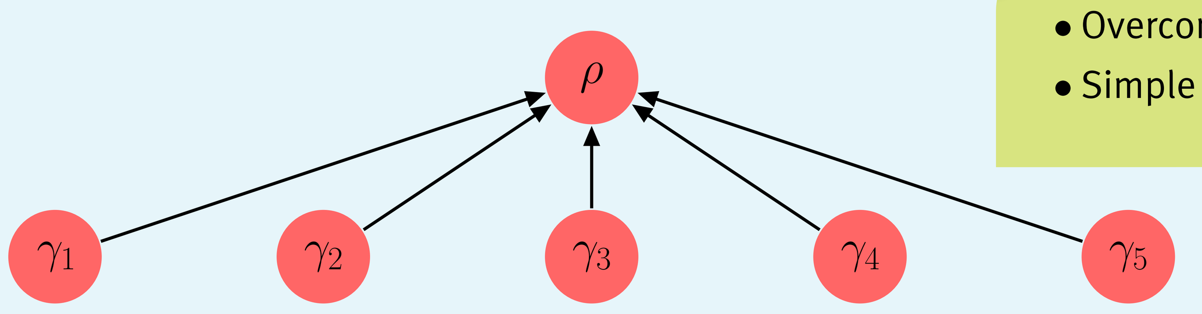
Mode Bound:

$$|\,\mathrm{HAPOD}(\mathcal{S}, \rho, \varepsilon)| \leq |\,\mathrm{POD}(\mathcal{S}, (1 - 2\omega) \cdot \varepsilon^*)|$$

Notation:

$\mathcal{S}_\alpha$: snapshots below α
M_α: POD modes at α
L: depth of tree
$L(\alpha)$: depth of α
P: projection onto HAPOD mode space at ρ
ω: parameter $\in (0, 1/2)$

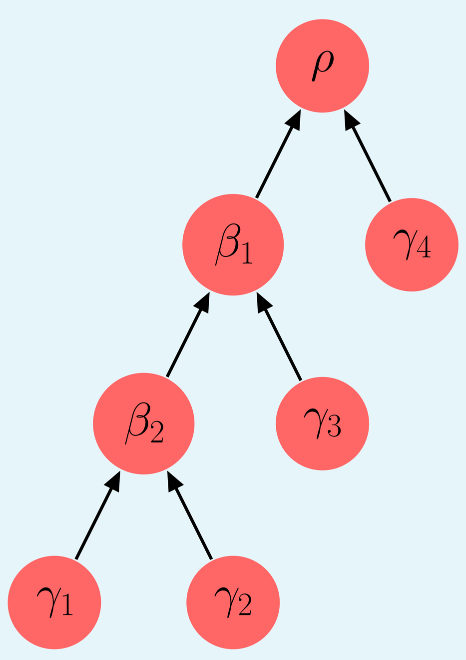
DAPOD – Distributed Approximate POD



- Overcome your RAM limitations!
- Simple parallelization!

Choice of Error Tolerances: $\varepsilon(\rho) = \frac{\sqrt{|S|}}{\sqrt{|M_\rho|}} \cdot (1 - \omega) \cdot \varepsilon^*, \quad \varepsilon(\gamma_i) = \omega \cdot \varepsilon^*$

RAPOD – Rolling Approximate POD



Choice of Error Tolerances:

$$\varepsilon(\rho) = \frac{\sqrt{|S|}}{\sqrt{|M_\rho|}} \cdot (1 - \omega) \cdot \varepsilon^* \qquad \varepsilon(\beta_i) = \frac{\sqrt{|S|}}{\sqrt{|M_{\beta_i}| \cdot (L-1)}} \cdot \omega \cdot \varepsilon^*$$

$$\varepsilon(\gamma_1) = \frac{1}{L-1} \cdot \omega \cdot \varepsilon^* \qquad \varepsilon(\gamma_i) = 0, \quad i > 1$$

- On-the-fly live data compression!
- Fast even when whole trajectory does not fit into RAM!

Benchmark Problem

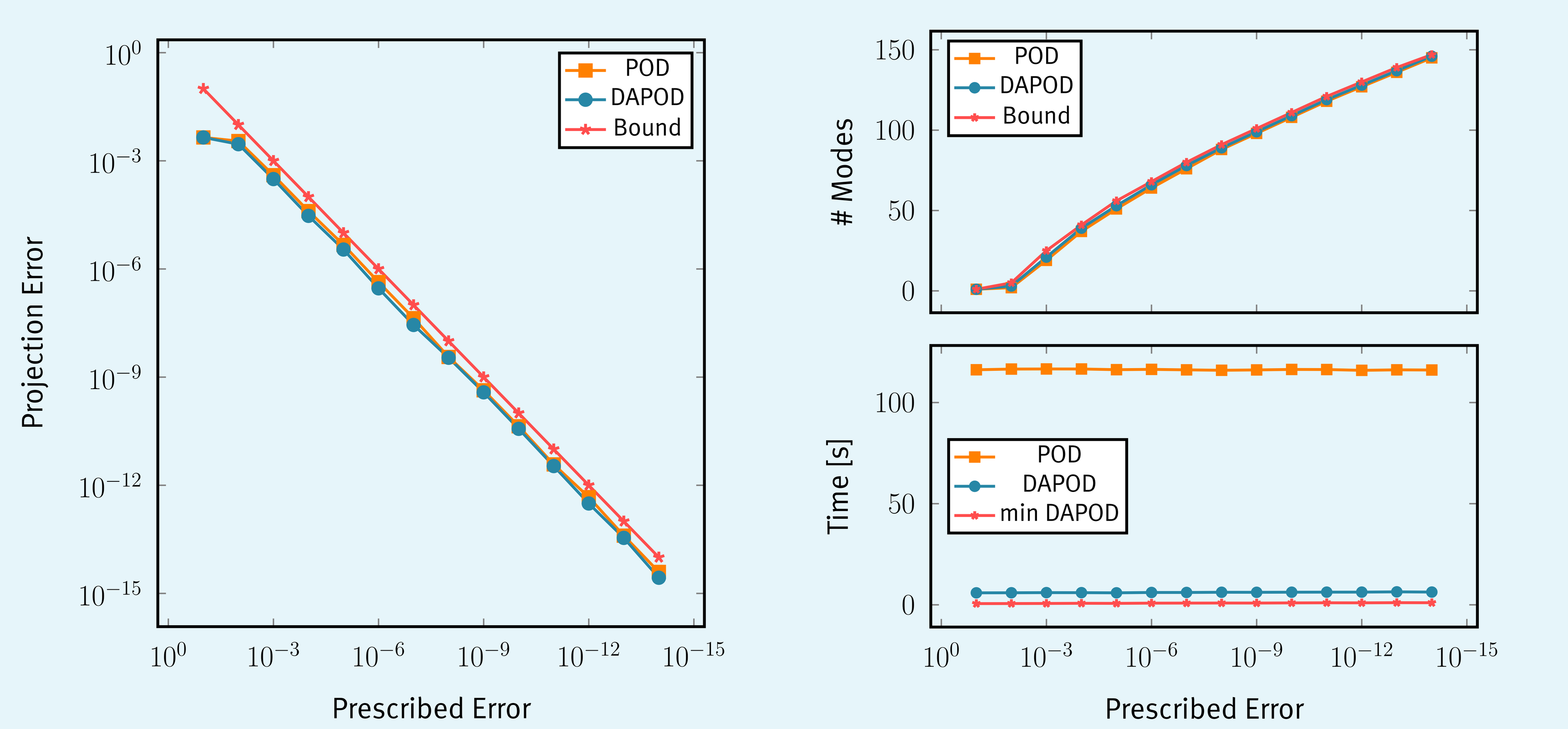
Synthetic Parametric Model (See MORwiki: `http://modelreduction.org/index.php/Synthetic_parametric_model`)

- linear time-invariant
- single-input
- affine-parametric
- $\dim(x(t)) := 10^5$
- $C := 1$
- $u(t) := \delta(t)$
- $\theta \in (0, 1]$
- implicit RK1
- $\mathcal{S}_{\text{DAPOD}} = \{x(t; \theta) \mid \theta = k \cdot 0.1, \ 1 \leq k \leq 10\}$
- $\mathcal{S}_{\text{RAPOD}} = \{x(t; \theta) \mid \theta = 0.5\}$

Get the Code

http://j.mp/morml16

DAPOD Numerical Experiment



RAPOD Numerical Experiment