Westfälische Wilhelms-Universität Münster

Applied Mathematics Münster

Combined Reduction of Hierarchical Systems

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Overview

Control Systems

Linear control system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with:

- Input $u(t) \in \mathbb{R}^m$
- State $x(t) \in \mathbb{R}^n$
- Output $y(t) \in \mathbb{R}^o$

Control Systems

Parametrized linear control system:

$$\dot{x}(t) = A_\theta x(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with:

- Input $u(t) \in \mathbb{R}^m$
- State $x(t) \in \mathbb{R}^n$
- Output $y(t) \in \mathbb{R}^o$
- Parameters $\theta \in \mathbb{R}^p$

Control Systems

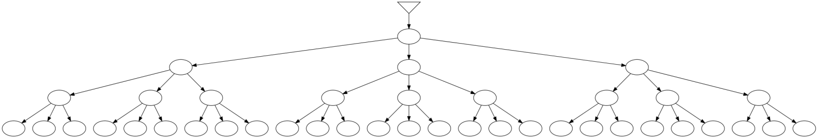
General control system:

$$\dot{x}(t) = f(x(t), u(t), \theta)$$
$$y(t) = g(x(t), u(t), \theta)$$

with:

- Input $u(t) \in \mathbb{R}^m$
- State $x(t) \in \mathbb{R}^n$
- Output $y(t) \in \mathbb{R}^o$
- Parameters $\theta \in \mathbb{R}^p$

Hierarchical Network



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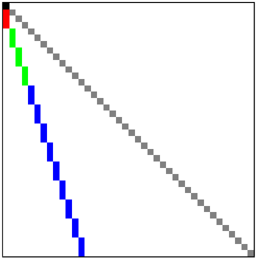
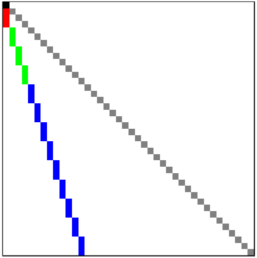
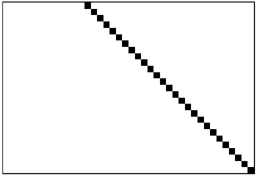
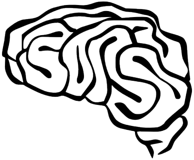


SIMO System

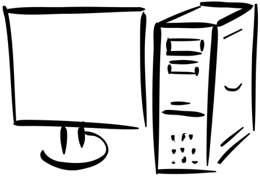
$\dot{x}(t) =$  $x(t) +$  $u(t)$

$y(t) =$  $x(t)$

Application



Neural Networks



Model Reduction

- High-dim state space $\rightarrow$ state reduction
- High-dim parameter space $\rightarrow$ parameter reduction
- High-dim state and parameter space $\rightarrow$ combined reduction

Gramian-Based State Reduction [Moore'81]

- Controllability Gramian
- Observability Gramian
- Balancing Transformation + Truncation

Gramian-Based Linear State Reduction [Moore'81]

- Controllability Gramian
- Controllability Gramian of Adjoint System[1]
- Balancing Transformation + Truncation

[1] Adjoint System: $\dot{x}(t) = A^T x(t) + C^T u(t)\ ; \quad y(t) = B^T x(t)$

Empirical Gramians [Lall'99]

- Correspond to analytical gramians for linear systems.
- Extend to nonlinear systems.
- Utilize solely basic matrix and vector operations.

Empirical Controllability Gramian

Concept:

- $W_C = \underset{u \in U}{mean}(\int_0^\infty x_u(t) x_u^*(t) dt)$

$U = E_u \times R_u \times Q_u$

- $E_u = \{e_i \in \mathbb{R}^j; \|e_i\| = 1; e_i e_{j \neq i} = 0; i = 1, \dots, m\}$
- $R_u = \{S_i \in \mathbb{R}^{j \times j}; S_i^* S_i = \mathbb{1}; i = 1, \dots, s\}$
- $Q_u = \{c_i \in \mathbb{R}; c_i > 0; i = 1, \dots, q\}$

Discrete Empirical Controllability Gramian [Hahn'02]

For sets E_u, R_u, Q_u, input $u(t)$ and input during the steady state $\bar{x}$, $\bar{u}$, the **discrete empirical controllability gramian** is given by:

$$W_C = \frac{1}{|Q_u||R_u|} \sum_{h=1}^{|Q_u|} \sum_{i=1}^{|R_u|} \sum_{j=1}^{m} \frac{\Delta t}{c_h^2} \sum_{t=0}^{T} \Psi^{hij}(t)$$

$$\Psi^{hij}(t) = (x^{hij}(t) - \bar{x})(x^{hij}(t) - \bar{x})^* \in \mathbb{R}^{n \times n}.$$

With x^{hij} being the states for the input configuration $u^{hij}(t) = c_h S_i e_j u(t) + \bar{u}$.

State Reduction

Recipe:

1. Empirical Controllability Gramian
2. Empirical Controllability Gramian of Adjoint System
3. Balancing Transformation + Truncation

Parameter Reduction

Recipe:

1. ?
2. Sensitivity Analysis
3. Parameter Truncation

Options:

- Sensitivity Gramian (Controllability-based)?
- Identifiability Gramian (Observability-based)?
- Joint Gramian (Cross-gramian-based)?

Parameter Reduction

Recipe:

1. Empirical Sensitivity Gramian
2. Sensitivity Analysis
3. Parameter Truncation

Options:

- Sensitivity Gramian (Controllability-based)? Yes!
- Identifiability Gramian (Observability-based)? No!
- Joint Gramian (Cross-gramian-based)? No!

Sensitivity Gramian (based on [Sun'06])

Treating the parameters as additional inputs of *dim*(θ) with steady input θ gives (if possible):

$$\dot{x} = f(x, u) + \sum_{k=1}^{P} f(x, \theta_k)$$
$$\Rightarrow W_C = W_{C,0} + \sum_{k=1}^{P} W_{C,k}$$

Sensitivity Gramian W_S:

$$W_{S,ii} = \operatorname{tr}(W_{C,i}).$$

(Linear) Combined Reduction

Recipe:

1. State Reduction
   1. Empirical Controllability Gramian
   2. Empirical Controllability Gramian of Adjoint System
   3. Balanced Truncation
2. Parameter Reduction
   1. Empirical Sensitivity Gramian
   2. Sensitivity Analysis
   3. Parameter Truncation

Implementation: Empirical Gramian Framework

Gramians:

- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

Features:

- Uniform Interface
- Compatible with MATLAB & OCTAVE
- Vectorized & Parallelizable
- Open-Source licensed

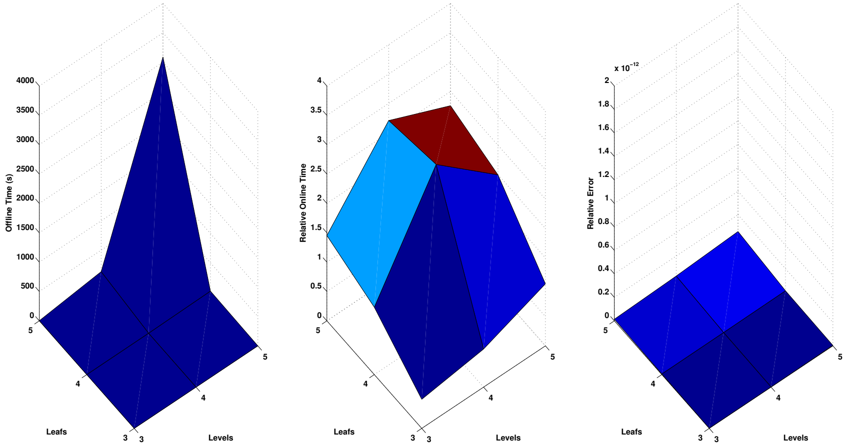
More info at: `http://gramian.de`

Numerical Experiments

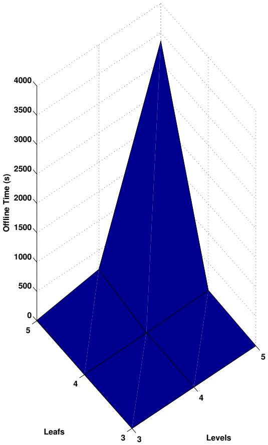
$$\dot{x}(t) = A_\theta x(t) + Bu(t)$$
$$y(t) = Cx(t)$$

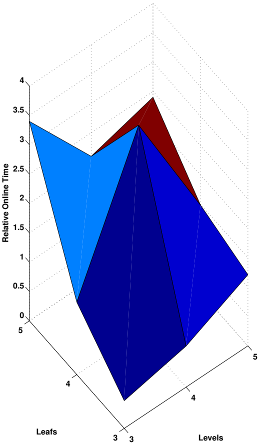
- for L-level M-ary trees
- with $L, M \in \{3, 4, 5\}$
- since $n = \dim(x) = \frac{M^{L+1}-1}{M-1}$
- $40 \leq \dim(x) \leq 3906$
- thus $p = \dim(\theta) = \dim(x)$

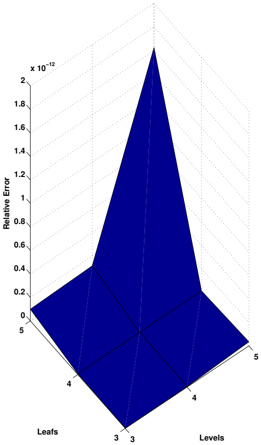
State Reduction



Combined Reduction







Results

- Sparsity can be exploited in snapshot generation.
- Uncertainties can be incorporated by perturbations of inputs.
- Especially useful for inverse problems.
- Only the Controllability Gramian is required.

The reduced order of states and parameters arose to: $\#\text{Levels}+1$.

tl;dl

- Combined Reduction: reduction of states and parameters.
- Empirical Gramians: reduced order nonlinear control systems.
- Hierarchical Systems: exploiting sparsity.
- Get the source code: `http://j.mp/enumath13` .

Thanks!