Westfälische Wilhelms-Universität Münster

Applied Mathematics Münster

Combined Reduction for EEG Model Inversion

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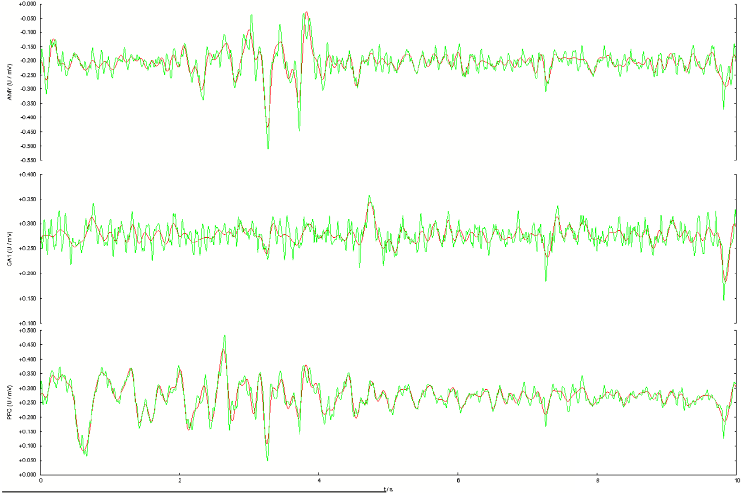
Motivation

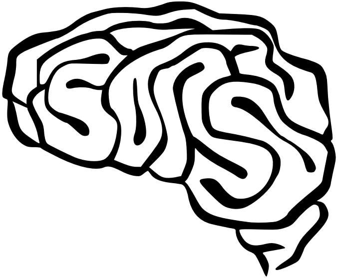
Intracranial EEG and tuned model output[1]:



[1] [Himpe'11]

Dynamic Causal Modelling

- How are brain regions connected?
- How does sensory input disperse?
- How does the brain learn and unlearn?



Dynamic Causal Modelling

- How are brain regions connected?
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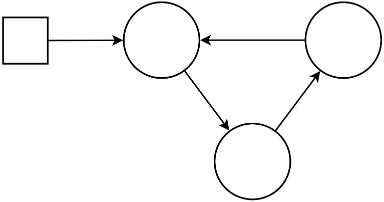
- Parametrized Dynamical System



Dynamic Causal Modelling

- How are brain regions connected?
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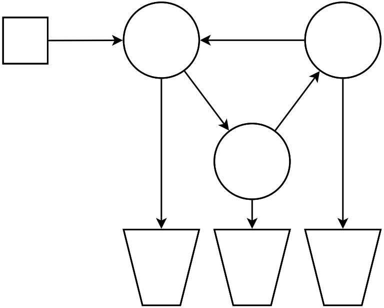
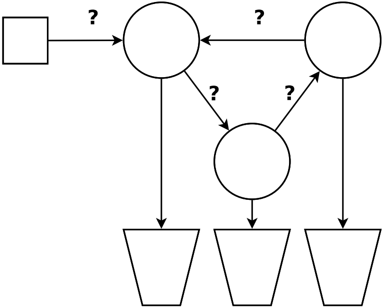
- Parametrized Dynamical System
- Output System



Dynamic Causal Modelling

- How are brain regions connected?
- How does sensory input disperse?
- How does the brain learn and unlearn?

- Parametrized Dynamical System
- Output System
- Bayesian Inference



Jansen Neural Mass Model

Synaptic Input: $u(t)$
Impulse Response: $h(t) = H\kappa t \exp(-t\kappa)$

Postsynaptic Membrane Potential:

$$
\begin{aligned}
& v(t) = h(t) * u(t) \\
\Rightarrow \quad & \ddot{v}(t) = H\kappa u(t) - 2\kappa\dot{v}(t) - \kappa^2 v(t) \\
\Rightarrow & \begin{cases} \dot{v}(t) = x(t) \\ \dot{x}(t) = H\kappa u(t) - 2\kappa x(t) - \kappa^2 v(t) \end{cases}
\end{aligned}
$$

Jansen Neural Mass Model

Synaptic Input: $u(t) = S(v(t)) + u_x(t)$
Impulse Response: $h(t) = H\kappa t \exp(-t\kappa)$

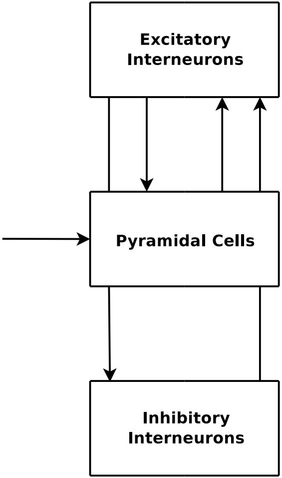
Postsynaptic Membrane Potential:

$$v(t) = h(t) * u(t)$$

$$\Rightarrow \quad \ddot{v}(t) = H\kappa u(t) - 2\kappa\dot{v}(t) - \kappa^2 v(t)$$

$$\Rightarrow \begin{cases} \dot{v}(t) = x(t) \\ \dot{x}(t) = H\kappa u(t) - 2\kappa x(t) - \kappa^2 v(t) \end{cases}$$

3-Layer Model[2]



$$v_0(t) = h_e(t) * u(v_2)$$
$$v_1(t) = h_i(t) * u(v_3, v_4)$$

$$v_2(t) = h_e(t) * u(v_0, v_1, u_x)$$

$$v_3(t) = h_e(t) * u(v_0, v_1)$$
$$v_4(t) = h_i(t) * u(v_3, v_4)$$

[2] see [Moran'07]

Single Region[3] (SISO)

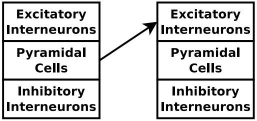
$$
\begin{aligned}
\dot{x}_0 &= x_5 \\
\dot{x}_1 &= x_6 \\
\dot{x}_2 &= x_7 \\
\dot{x}_3 &= x_8 \\
\dot{x}_4 &= x_9 \\
\dot{x}_5 &= \kappa_e H_e \gamma_2 S(x_2) \qquad\quad - 2\kappa_e x_5 - \kappa_e^2 x_0 \\
\dot{x}_6 &= \kappa_i H_i \gamma_4 S(x_3 - x_4) \ - 2\kappa_i x_6 - \kappa_i^2 x_1 \\
\dot{x}_7 &= \kappa_e H_e \gamma_1 S(x_0 - x_1) - 2\kappa_e x_7 - \kappa_e^2 x_2 + \kappa_e H_e \gamma_1 u \\
\dot{x}_8 &= \kappa_e H_e \gamma_3 S(x_0 - x_1) - 2\kappa_e x_8 - \kappa_e^2 x_3 \\
\dot{x}_9 &= \kappa_i H_i \gamma_5 S(x_3 - x_4) \ - 2\kappa_i x_9 - \kappa_i^2 x_4
\end{aligned}
$$

$$
y = x_0 - x_1
$$

[3] see [Moran'07]

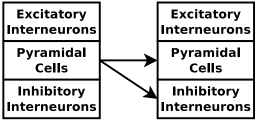
Connectivity[4]

Forward Connection:



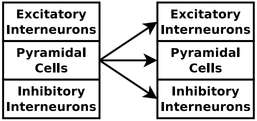
A_F

Backward Connection:



A_B

Lateral Connection:



A_L

[4] [David'04]

Multiple Regions[5] (MIMO)

$$\begin{aligned}
\dot{X}_0 &= X_5 \\
\dot{X}_1 &= X_6 \\
\dot{X}_2 &= X_7 \\
\dot{X}_3 &= X_8 \\
\dot{X}_4 &= X_9 \\
\dot{X}_5 &= \kappa_e H_e (A_B + A_L + \gamma_2 \mathbb{1}) S(X_2) \qquad - 2\kappa_e X_5 - \kappa_e^2 X_0 \\
\dot{X}_6 &= \qquad \kappa_i H_i \gamma_4 \mathbb{1}\; S(X_3 - X_4) - 2\kappa_i X_6 - \kappa_i^2 X_1 \\
\dot{X}_7 &= \kappa_e H_e (A_F + A_L + \gamma_1 \mathbb{1}) S(X_0 - X_1) - 2\kappa_e X_7 - \kappa_e^2 X_2 + \kappa_e H_e \gamma_1 U \\
\dot{X}_8 &= \kappa_e H_e (A_B + A_L + \gamma_3 \mathbb{1}) S(X_0 - X_1) - 2\kappa_e X_8 - \kappa_e^2 X_3 \\
\dot{X}_9 &= \qquad \kappa_i H_i \gamma_5 \mathbb{1}\; S(X_3 - X_4) - 2\kappa_i X_9 - \kappa_i^2 X_4
\end{aligned}$$

$$Y = C(X_0 - X_1)$$

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[5] see [David'06]

Control System

Linear Control System:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
$$

General Control System:

$$
\begin{aligned}
\dot{x} &= f(x, u, \theta) \\
y &= g(x, u, \theta)
\end{aligned}
$$

DCM-EEG **Neural Mass Model**:

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & \mathbb{1} \\ -\kappa^2\,\mathbb{1} & -2\kappa\,\mathbb{1} \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ \kappa H \Sigma(\theta) & 0 \end{pmatrix} S(x) + Bu \\
y &= Cx
\end{aligned}
$$

$\rightarrow$ Nonlinear Second-Order Control System!

(Nonlinear) Model Reduction

State-Space Reduction:

- (Empirical) Gramian-Based
  1. Balanced Truncation
  2. Approximate Balancing (Cross Gramian)

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Parameter-Space Reduction:

- (Empirical) Gramian-Based
  1. Controllability-Based (parameters as additional inputs)
  2. Observability-Based (parameters as additional states)
  3. Cross-Gramian-Based (parameters as additional states)

(Nonlinear) Model Reduction

State-Space Reduction:

- (Empirical) Gramian-Based
  1. Balanced Truncation
  2. Approximate Balancing (Cross Gramian)

Parameter-Space Reduction:

- (Empirical) Gramian-Based
  1. Controllability-Based (parameters as additional inputs)
  2. Observability-Based (parameters as additional states)
  3. Cross-Gramian-Based (parameters as additional states)

Combined (State and Parameter) Reduction:

- (Empirical) Gramian-Based
  1. Controllability-Based
  2. Observability-Based
  3. Cross-Gramian-Based

Second-Order Balanced Truncation[6]

Second-Order System Gramians:

Controllability Gramian: Observability Gramian: Cross Gramian:

$$W_C = \begin{pmatrix} W_{C,P} & W_{C,PV} \\ W_{C,VP} & W_{C,V} \end{pmatrix} \qquad W_O = \begin{pmatrix} W_{O,P} & W_{O,PV} \\ W_{O,VP} & W_{O,V} \end{pmatrix} \qquad W_X = \begin{pmatrix} W_{X,P} & W_{X,PV} \\ W_{X,VP} & W_{X,V} \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Position Gramian: } W_{?,P} \\ \text{Velocity Gramian: } W_{?,V} \end{array}\right\} \rightarrow \{U_P, U_V, V_P, V_V\}$$

Second-Order Projections:

$$\left.\begin{array}{l} \dot{x} = \begin{pmatrix} x_V \\ f(\begin{pmatrix} x_P \\ x_V \end{pmatrix}, u, \theta) \end{pmatrix} \\ y = \quad g(\begin{pmatrix} x_P \\ x_V \end{pmatrix}, u, \theta) \end{array}\right\} \Rightarrow \begin{cases} \dot{\tilde{x}} = \begin{pmatrix} \tilde{x}_V \\ V_V f(\begin{pmatrix} U_P \tilde{x}_P \\ U_V \tilde{x}_V \end{pmatrix}, u, \theta) \end{pmatrix} \\ \tilde{y} = \qquad g(\begin{pmatrix} U_P \tilde{x}_P \\ U_V \tilde{x}_V \end{pmatrix}, u, \theta) \end{cases}$$

[6] see [Reis'07]

A zoo of methods[7]

Balanced Decomposition:

$$BAL(W_{C,P}, W_{O,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$
$$BAL(W_{C,V}, W_{O,V}) \rightarrow \{U_V, V_V\} \equiv \{U_P, V_P\}$$
$$BAL(W_{C,P}, W_{O,V}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$
$$BAL(W_{C,V}, W_{O,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$
$$\left.\begin{array}{l} BAL(W_{C,P}, W_{O,P}) \\ BAL(W_{C,V}, W_{O,V}) \end{array}\right\} \rightarrow \{U_P, V_P, U_V, V_V\}$$

Singular Value Decomposition:

$$SVD(W_{C,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$
$$SVD(W_{C,V}) \rightarrow \{U_V, V_V\} \equiv \{U_P, V_P\}$$
$$SVD(W_{O,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$
$$SVD(W_{O,V}) \rightarrow \{U_V, V_V\} \equiv \{U_P, V_P\}$$
$$\left.\begin{array}{l} SVD(W_{C,P}) \\ SVD(W_{C,V}) \end{array}\right\} \rightarrow \{U_P, V_P, U_V, V_V\}$$
$$\left.\begin{array}{l} SVD(W_{O,P}) \\ SVD(W_{O,V}) \end{array}\right\} \rightarrow \{U_P, V_P, U_V, V_V\}$$

$$SVD(W_{X,P}) \rightarrow \{U_P, V_P\} \equiv \{U_V, V_V\}$$
$$SVD(W_{X,V}) \rightarrow \{U_V, V_V\} \equiv \{U_P, V_P\}$$
$$\left.\begin{array}{l} SVD(W_{X,P}) \\ SVD(W_{X,V}) \end{array}\right\} \rightarrow \{U_P, V_P, U_V, V_V\}$$

7 [Teng'12]

Empirical Gramians[8] I

POD-based method:

- Empirical Controllability Gramian: $W_C = \langle \int_0^\infty x_U(t) x_U^*(t) dt \rangle_U$
- Empirical Observability Gramian: $W_O = \langle \int_0^\infty \rho(y_X^*(t) y_X(t)) dt \rangle_X$
- Empirical Cross Gramian: $W_X = \langle \int_0^\infty \varphi(x_U(t), y_X(t)) dt \rangle_{U \times X}$

with perturbation spaces:

1. U perturbing the input u,
2. X perturbing the initial state x_0

assembled from:

- rotations, sets of orthogonal matrices Q_u, Q_x
- scales, sets of real numbers R_u, R_x
- for each input / state, the unit normal vectors E_u, E_x

determined by the operating range of the underlying contol system.

[8] see [Lall'99]

Empirical Gramians[9] II

Empirical Controllability Gramian:

$$W_C = \frac{1}{|Q_u||R_u|} \sum_{h=1}^{|Q_u|} \sum_{i=1}^{|R_u|} \sum_{j=1}^{m} \frac{1}{c_h^2} \int_0^\infty \Psi^{hij}(t) dt$$

$$\Psi^{hij}(t) = (x^{hij}(t) - \bar{x})(x^{hij}(t) - \bar{x})^* \in \mathbb{R}^{n \times n},\ u^{hij}(t) = c_h S_i e_j u(t) + \bar{u}$$

Empirical Observability Gramian:

$$W_O = \frac{1}{|Q_x||R_x|} \sum_{k=1}^{|Q_x|} \sum_{l=1}^{|R_x|} \frac{1}{d_k^2} T_l \int_0^\infty \Psi^{kl}(t) dt\ T_l^*$$

$$\Psi_{ab}^{kl}(t) = (y^{kla}(t) - \bar{y})^*(y^{klb}(t) - \bar{y}) \in \mathbb{R},\ x_0^{kla} = d_k T_l f_a + \bar{x}$$

Empirical Cross Gramian:

$$W_X = \frac{1}{|Q_u||R_u|m|Q_x||R_x|} \sum_{h=1}^{|Q_u|} \sum_{i=1}^{|R_u|} \sum_{j=1}^{m} \sum_{k=1}^{|Q_x|} \sum_{l=1}^{|R_x|} \frac{1}{c_h d_k} \int_0^\infty T_l \Psi^{hijkl}(t) T_l^* dt$$

$$\Psi_{ab}^{hijkl}(t) = f_b^* T_k^* \Delta x^{hij}(t) e_i^* S_h^* \Delta y^{kla}(t) \in \mathbb{R}$$

$$\Delta x^{hij}(t) = (x^{hij}(t) - \bar{x}),\ u^{hij}(t) = c_h S_i e_j u(t) + \bar{u}$$

$$\Delta y^{kla}(t) = (y^{kla}(t) - \bar{y}),\ x_0^{kla} = d_k T_l f_a + \bar{x}$$

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9 [Hahn'02], [Streif'06], [Himpe'13a]

Empirical Gramians[10] III

Empirical Sensitivity Gramian (Controllability-Based) W_S :

$$\hat{u} = \begin{pmatrix} u \\ \theta \end{pmatrix} \rightarrow \dot{x} = f(x, \hat{u}) = f(x, u) + \sum_{k=1}^{P} f(x, \theta_k) \Rightarrow W_C = W_{C,0} + \sum_{k=1}^{P} W_{C,k}$$

$$\rightarrow W_S = \delta_{i,j} \operatorname{trace}(W_{C,i})$$

Empirical Identifiability Gramian (Observability-Based) W_I :

$$\hat{x} = \begin{pmatrix} x \\ \theta \end{pmatrix} \rightarrow \dot{\hat{x}} = f(\hat{x}, u) = \begin{pmatrix} f(x, u, \theta) \\ 0 \end{pmatrix}, \hat{x}(0) = \begin{pmatrix} x_0 \\ \theta \end{pmatrix} \Rightarrow \hat{W}_O = \left(\begin{array}{c|c} W_O & W_M \\ \hline W_M^* & W_P \end{array}\right)$$

$$\rightarrow W_I = W_P - W_M^* {W_O}^{-1} W_M \approx W_P$$

Empirical Joint Gramian (Cross-Gramian-Based) W_J :

$$\hat{x} = \begin{pmatrix} x \\ \theta \end{pmatrix} \Rightarrow W_J := \hat{W}_X = \left(\begin{array}{c|c} W_X & W_M \\ \hline 0 & 0 \end{array}\right)$$

$$\rightarrow W_{\ddot{I}} = -W_M^* (W_X + W_X^T)^{-1} W_M \approx -W_M^* \operatorname{diag}(W_X + W_X^T)^{-1} W_M$$

---

[10] [Sun'06], [Geffen'08], [Himpe'13a]

Combined Reduction

Controllability-Based

1. Compute W_S $\rightarrow W_C$
2. Decompose W_S
3. Truncate θ
4. Compute W_O
5. Balance W_C, W_O
6. Decompose W_{CO}
7. Truncate x

Observability-Based

1. Compute W_I $\rightarrow W_O$
2. Decompose W_I
3. Truncate θ
4. Compute W_C
5. Balance W_C, W_O
6. Decompose W_{CO}
7. Truncate x

Cross-Gramian-Based

1. Compute $W_{\bar{I}}$ $\rightarrow W_X$
2. Decompose $W_{\bar{I}}$
3. Truncate θ
4. Decompose W_X
5. Truncate x

emgr - Empirical Gramian Framework[11]

Gramians:

- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

Features:

- Uniform Interface
- Compatible with MATLAB & OCTAVE
- Vectorized & Parallelizable
- Open-Source licensed

More info at: `http://gramian.de`

[11] see [Himpe'13]

Numerical Experiments

Setup:

- Nonlinear DCM-EEG model
- Inverse problem on connectivity parameters: A_F, A_B, A_L
- Synthetic data for uniformly random connectivity
- with additive Gaussian noise: $y_d = y + N(0, v)$
- Combined reduction with: $\{W_S, W_O\}, \{W_C, W_I\}, W_J$
- Leapfrog integration

Dimensions:

- Regions: $n = \{4, \ldots, 10\}$
- States: $\dim(x) = N = 10n$
- Parameters: $\dim(\theta) = 3n^2$
- Reduced States: $\dim(\tilde{x}) = 4n$
- Reduced Parameters: $\dim(\tilde{\theta}) = 6n$

Combined Reduction for Inverse Problems

Inverse Problem:

- Bayesian inference: $P(\theta|y_d) \propto P(y_d|\theta)P(\theta)$
- Prior distribution: $A_{F,prior} = A_{B,priors} = A_{L,prior} = N(-\mathbb{1}, \mathbb{1})$
- Fixed biological parameters: $\kappa_e, H_e, \kappa_i, H_i$

Combined Reduction:

1. Offline Phase:
   Assembly of reduced order model using combined reduction.
2. Online Phase:
   Optimization reduced model using least-squares.

Applicability

Balanced Decomposition:

$BAL(W_{C,P}, W_{O,P})$	✓
$BAL(W_{C,V}, W_{O,V})$	✓
$BAL(W_{C,P}, W_{O,V})$	✗
$BAL(W_{C,V}, W_{O,P})$	✗
$\left.\begin{matrix} BAL(W_{C,P}, W_{O,P}) \\ BAL(W_{C,V}, W_{O,V}) \end{matrix}\right\}$	✗

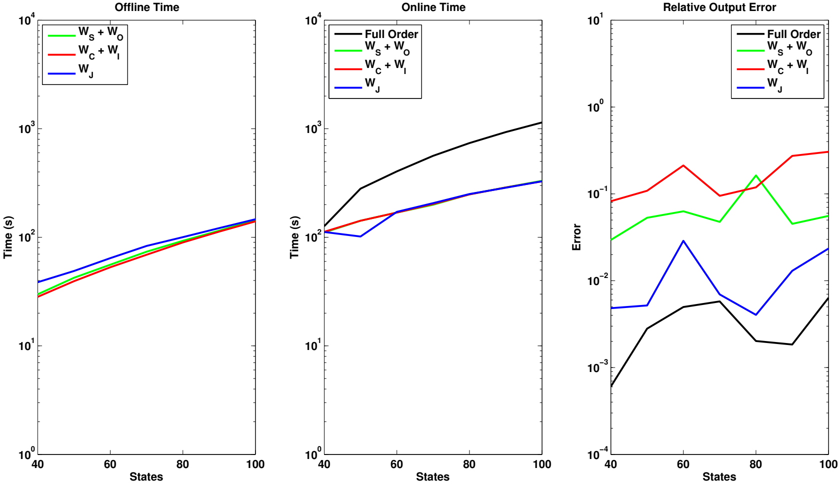
Singular Value Decomposition:

$SVD(W_{C,P})$	✗
$SVD(W_{C,V})$	✗
$SVD(W_{O,P})$	✗
$SVD(W_{O,V})$	✗
$\left.\begin{matrix} SVD(W_{C,P}) \\ SVD(W_{C,V}) \end{matrix}\right\}$	✗
$\left.\begin{matrix} SVD(W_{O,P}) \\ SVD(W_{O,V}) \end{matrix}\right\}$	✗
$SVD(W_{X,P})$	✓
$SVD(W_{X,V})$	✓
$\left.\begin{matrix} SVD(W_{X,P}) \\ SVD(W_{X,V}) \end{matrix}\right\}$	✗

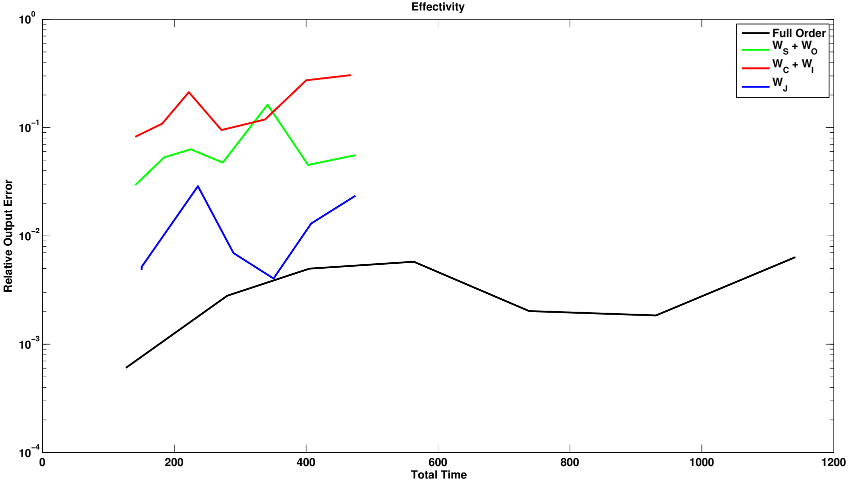
For experiments:

- $BAL(W_{C,P}, W_{O,P})$ for $\{W_S, W_O\}$
- $BAL(W_{C,P}, W_{O,P})$ for $\{W_C, W_I\}$
- $SVD(W_{X,P})$ for W_J

Numerical Results



Reduction Effectivity



tl;dl

Summary:

- Model Reduction prior to inversion,
- of Dynamic-Causal-EEG-Model.
- Nonlinear Model Reduction: Empirical Gramians.
- Combined Reduction: Reduction of States and Parameters.
- Cross-Gramian-Based Joint Gramian: very efficient.

- Get the Source Code: `http://j.mp/modred13` .

Outlook:

- Add delay elements
- Towards Neural-Field-Model

Thanks!