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POD, SVD, PCA, etc

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Overview

1. SVD
2. POD
3. PCA

SVD - Singular Value Decomposition

Given

- $X \in \mathbb{R}^{m \times n}$, $m \leq n$
- $\sigma_i = \sqrt{\lambda_i(XX^T)}$, $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_m$

then

$$X = UDV^T$$

with

- $U \in \mathbb{R}^{m \times m}$, $UU^T = \mathbb{1}$ (Left Singular Vectors)
- $D \in \mathbb{R}^{m \times n}$, $D_{ij} = \sigma_i \delta_{ij}$ (Singular Values)
- $V \in \mathbb{R}^{n \times n}$, $VV^T = \mathbb{1}$ (Right Singular Vectors)

POD - Proper Orthogonal Decomposition

Given

- $[x_1, \ldots, x_n] = X \in \mathbb{R}^{m \times n}$
- $R = \mathbb{E}[XX^T]$ (Autocorrelation Matrix)

then

$$Rw_i = \lambda_i w_i \overset{SVD}{\Rightarrow} X = UDV^T$$

with

- $U = [u_1, \ldots, u_m] = [w_1, \ldots, w_m]$ (POD Modes)
- $\lambda_i(R) = \frac{\sigma_i^2}{n}$, (Empirical Eigenvalues)

note

- also called KLT - Karhunen Loeve Transformation

PCA - Principal Component Analysis

Given

- $[x_1, \ldots, x_n] = X \in \mathbb{R}^{m \times n}$, $\frac{1}{n} \sum_{i=1}^{n} x_i = 0$
- $C = \mathbb{E}[(X - \mathbb{E}[X])(X - E[X])^T]$ (Covariance Matrix)

then

$$Cw_i = \lambda_i w_i \overset{SVD}{\Rightarrow} (X - \mathbb{E}[X]) = UDV^T$$

with

- $U = [u_1, \ldots, u_m] = [w_1, \ldots, w_m]$ (Principal Components)
- $\lambda_i(C) = \frac{\sigma_i^2}{n}$ (Weights)

note

- also called EOF - Empirical Orthogonal Functions

tl;dr

- $POD(X) \Leftrightarrow SVD(X)$
- $PCA(X) \Leftrightarrow SVD(X - \mathbb{E}[X])$