Westfälische Wilhelms-Universität Münster

Applied Mathematics Münster

emgr - Empirical Gramian Framework

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Overview

Control Systems

(in a hurry)

Control Systems

General control system:

$$\dot{x} = f(x, u, \theta)$$
$$y = g(x, u, \theta)$$

with:

- Input $u \in \mathbb{R}^m$
- State $x \in \mathbb{R}^n$
- Output $y \in \mathbb{R}^o$
- Parameters $\theta \in \mathbb{R}^p$

Control Systems

Linear control system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with:

- Input $u \in \mathbb{R}^m$
- State $x \in \mathbb{R}^n$
- Output $y \in \mathbb{R}^o$
- Parameters $\theta \in \mathbb{R}^p$

Control Systems

Linear control system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with:

- Input $u \in \mathbb{R}^m$
- State $x \in \mathbb{R}^n$
- Output $y \in \mathbb{R}^o$
- Parameters $\theta \in \mathbb{R}^p$

Control Systems

Nonlinear control system:

$$\dot{x} = A\tanh(x) + Bu$$
$$y = Cx$$

with:

- Input $u \in \mathbb{R}^m$
- State $x \in \mathbb{R}^n$
- Output $y \in \mathbb{R}^o$
- Parameters $\theta \in \mathbb{R}^p$

Control Systems

Parametrized linear control system:

$$\dot{x} = A_\theta x + Bu$$
$$y = Cx$$

with:

- Input $u \in \mathbb{R}^m$
- State $x \in \mathbb{R}^n$
- Output $y \in \mathbb{R}^o$
- Parameters $\theta \in \mathbb{R}^p$

Control Theory Vocabulary

- Linear System: $\Sigma = (A, B, C)$
- Adjoint System: $\Sigma^T = (A^T, C^T, B^T)$
- Markov parameter: $m(k) = CA^k B$
- Impulse response: $h(t) = Ce^{At} B$
- Input-to-state map: $x(t) = e^{At} Bu(t)$
- State-to-output map: $y(t) = Ce^{At} x_0$
- Transfer function: $G(s) = C(s\mathbb{1} - A)^{-1} B$
- System gain: $S = tr(CA^{-1} B) = G(0)$

Model Order Reduction

General Control System:

$$\dot{x} = f(x, u, \theta)$$
$$y = g(x, u, \theta)$$

with:

- $\dim(x) >> 1$
- $\dim(\theta) >> 1$

Model Order Reduction

State Reduced System:

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, u, \theta)$$
$$\tilde{y} = \tilde{g}(\tilde{x}, u, \theta)$$

with:

- $\dim(x) >> 1$
- $\dim(\tilde{x}) < \dim(x)$
- $\tilde{y} \approx y$

Model Order Reduction

Parameter Reduced System:

$$\dot{\tilde{x}} = \tilde{f}(x, u, \tilde{\theta})$$
$$\tilde{y} = \tilde{g}(x, u, \tilde{\theta})$$

with:

- $\dim(\theta) >> 1$
- $\dim(\tilde{\theta}) < \dim(\theta)$
- $\tilde{y} \approx y$

Model Order Reduction

Combined Reduced System:

$$\dot{\tilde{x}} = \tilde{\tilde{f}}(\tilde{x}, u, \tilde{\theta})$$
$$\tilde{y} = \tilde{\tilde{g}}(\tilde{x}, u, \tilde{\theta})$$

with:

- $\dim(x) >> 1$
- $\dim(\theta) >> 1$
- $\dim(\tilde{x}) < \dim(x)$
- $\dim(\tilde{\theta}) < \dim(\theta)$
- $\tilde{y} \approx y$

Gramian-Based Model Reduction

(and System Identification)

Hankel Operator

- For a linear control system $\Sigma = (A, B, C)$.
- The Hankel operator $H(t)$ maps past inputs to future outputs
- using the impulse response

$$H(t) = \int_{-\infty}^{0} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)d\tau$$

- We are interested in the Singular Values of $H(t)$!

Controllability Gramian (W_C)

Controllability quantifies how well a state can be driven by input or control.

The controllability gramian W_C characterizes the controllability of a linear system and is computed as the smallest semi-positive definite solution of the Lyapunov equation:

$$AW_C + W_C A^T = -BB^T.$$

For asymptotic stable systems W_C is also given by:

$$W_C = \int_0^\infty e^{At} BB^T e^{A^T t} dt.$$

Observability Gramian (W_O)

Observability quantifies how well a change in state is reflected by the outputs.

The observability gramian W_O characterizes the observability of a linear system and is computed as the smallest semi-positive definite solution of the Lyapunov equation:

$$A^T W_O + W_O A = -C^T C.$$

For asymptotic stable systems W_O is also given by:

$$W_O = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$$

Balanced Truncation[1] (BT)

Hankel Singular Values:

$$\sigma_i = \sqrt{\lambda(W_C W_O)}$$

Balancing Transformation:

$$\exists U, V : V W_C V^T = U^T W_O U = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}$$

thus:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

1 [Moore'81]

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thus:

$$\dot{x} = VAUx + VBu$$
$$y = CUx$$

---

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thus:

$$\dot{x} = \tilde{A}x + \tilde{B}u$$
$$y = \tilde{C}x$$

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thus:

$$\dot{x} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u$$

$$y = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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thus:

$$\dot{\tilde{x}} = \tilde{A}_{11} \tilde{x}_1 + \tilde{B}_1 u$$
$$\tilde{y} = \tilde{C}_1 \tilde{x}_1$$

[1] [Moore'81]

Cross Gramian (W_X)

The cross gramian characterizes both, the controllability and the observability of a linear system.

Given the system has the same number of inputs and outputs, the cross gramian is computed as the smallest solution of the Sylvester equation:

$$AW_X + W_X A = -BC$$

For asymptotic stable systems W_X is also given by:

$$W_X = \int_0^\infty e^{At} B C e^{At} dt$$

Direct Truncation (Approximate Balanced Reduction[2])

For SISO or symmetric MIMO (has symmetric transfer function) systems (A, B, C) there exists a J such that:

$$J = J^T, \quad AJ = JA^T, \quad B^T = CJ$$

then:

$$W_O W_C = W_X^2 \Rightarrow \sigma_i = |\lambda(W_X)| \Rightarrow W_X = UDV \approx U \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}$$

thus:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

[2] [Sorensen'01], [Sorensen'02]

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thus:

$$\dot{x} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u$$

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thus:

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}_{11}\tilde{x}_1 + \tilde{B}_1 u \\ \tilde{y} &= \tilde{C}_1 \tilde{x}_1 \end{aligned}$$

[2] [Sorensen'01], [Sorensen'02]

Nonsymmetric Cross Gramian

- Symmetricity is strong requirement
- For nonsymmetric systems only

$$\sum_{i=1}^{k} \sigma_i \geq \sum_{i=1}^{k} \lambda(W_X)_i \wedge \sum_{i=1}^{k} \lambda(W_X)_{N-i+1} \geq \sum_{i=1}^{k} \sigma_{N-i+1}$$

  holds.

But:

- embed into symmetric system
- orthogonal symmetricity
- use approximate nonsymmetric Cross Gramian ⚠

Sensitivity Analysis

- Given a parametrized system, with many parameters...
- Which parameters are important? Which are redundant?
  $\Rightarrow$ Parameter Identification
- For a generic model, which parameters can be excluded here?
  $\Rightarrow$ Parameter Reduction

- So what is the dominant subspace of the parameter space?
- Wait a minute, that sounds familiar...

Sensitivity Gramian[3] (W_S)

Treating the parameters as additional inputs of $dim(\theta)$ with steady input θ (if possible) gives:

$$\dot{x} = f(x,u) + \sum_{k=1}^{P} f(x,\theta_k)$$

$$\Rightarrow W_C = W_{C,0} + \sum_{k=1}^{P} W_{C,k}$$

Sensitivity Gramian W_S:

$$W_{S,ii} = \mathrm{tr}(W_{C,i}).$$

The state controllability gramian W_C is a byproduct!

[3] based on [Sun'06]

Identifiability Gramian[4] (W_I)

Augmenting the system by $\dim(\theta)$ constant states with initial value being θ gives:

$$\dot{x}_a = \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} f(x, u, \theta) \\ 0 \end{pmatrix}$$

$$x_a(0) = \begin{pmatrix} x_0 \\ \theta \end{pmatrix}$$

$$\Rightarrow W_{O,a} = \left(\begin{array}{c|c} W_O & W_M \\ \hline W_M^* & W_P \end{array}\right).$$

Identifiability Gramian W_I:

$$W_I = W_P - W_M^* W_O{}^{-1} W_M,$$
$$W_I \approx W_P.$$

The state observability gramian W_O is a byproduct!

[4] [Geffen'08]

Combined Reduction

- High-dim state space $\rightarrow$ state reduction
- High-dim parameter space $\rightarrow$ parameter reduction
- High-dim state and parameter spaces $\rightarrow$ combined reduction

Joint Gramian[5] (W_J)

Augmenting the system by $\dim(\theta)$ constant states with initial value being θ:

$$\dot{x}_a = \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} f(x, u, \theta) \\ 0 \end{pmatrix}$$
$$x_a(0) = \begin{pmatrix} x_0 \\ \theta \end{pmatrix}$$
$$\Rightarrow W_{J,a} = \left(\begin{array}{c|c} W_X & W_M \\ \hline 0 & 0 \end{array}\right).$$

Cross-Identifiability Gramian W_{II}:

$$W_{II} = -W_M^* \,\mathrm{diag}(W_X + W_X^T)^{-1} W_M.$$

~~The state cross gramian W_X is~~ a byproduct!

[5] [Himpe'13]

Empirical Gramian-Based Model Reduction

(going nonlinear)

Empirical Gramian[6] / Empirical Covariance Matrix[7]

Concept: POD Method

- $W_C = \underset{u \in U}{mean}(\int_0^\infty x(t)x^*(t)dt)$
- $W_O = \underset{x_0 \in X}{mean}(\int_0^\infty \rho(y^*(t)y(t))dt)$
- $W_X = \underset{u \in U, x_0 \in X}{mean}(\int_0^\infty \varphi(x(t), y(t))dt)$

with

- U some perturbation to the input u
- X some perturbation to the initial state x_0

Empircal Covariance Matrices correspond to the Empirical Gramians, but allow arbitrary input instead of delta-impulses.

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[6] [Lall'99],[Lall'02]

[7] [Hahn'02]

Perturbation Sets

$U = E_u \times R_u \times Q_u$

- $E_u = \{e_i \in \mathbb{R}^j; \|e_i\| = 1; e_i e_{j \neq i} = 0; i = 1, \ldots, m\}$
- $R_u = \{S_i \in \mathbb{R}^{j \times j}; S_i^* S_i = \mathbb{1}; i = 1, \ldots, s\}$
- $Q_u = \{c_i \in \mathbb{R}; c_i > 0; i = 1, \ldots, q\}$

$X = E_x \times R_x \times Q_x$

- $E_x = \{f_i \in \mathbb{R}^n; \|f_i\| = 1; f_i f_{j \neq i} = 0; i = 1, \ldots, n\}$
- $R_x = \{T_i \in \mathbb{R}^{n \times n}; T_i^* T_i = \mathbb{1}; i = 1, \ldots, t\}$
- $Q_x = \{d_i \in \mathbb{R}; d_i > 0; i = 1, \ldots, r\}$

Empirical Controllability Gramian (WC)

For sets E_u, R_u, Q_u, input $u(t)$ and input during the steady state $\bar{x}$, $\bar{u}$, the **empirical controllability gramian** is given by:

$$W_C = \frac{1}{|Q_u||R_u|} \sum_{h=1}^{|Q_u|} \sum_{i=1}^{|R_u|} \sum_{j=1}^{m} \frac{1}{c_h^2} \int_0^\infty \Psi^{hij}(t) dt$$

$$\Psi^{hij}(t) = (x^{hij}(t) - \bar{x})(x^{hij}(t) - \bar{x})^* \in \mathbb{R}^{n \times n}.$$

With x^{hij} being the states for the input configuration $u^{hij}(t) = c_h S_i e_j u(t) + \bar{u}$.

Empirical Observability Gramian (WO)

For sets E_x, R_x, Q_x and output y during the steady state $\bar{x}$, $\bar{y}$, the **empirical observability gramian** is given by:

$$W_O = \frac{1}{|Q_x||R_x|} \sum_{k=1}^{|Q_x|} \sum_{l=1}^{|R_x|} \frac{1}{d_k^2} T_l \int_0^\infty \Psi^{kl}(t) dt \ T_l^*$$

$$\Psi_{ab}^{kl} = (y^{kla}(t) - \bar{y})^* (y^{klb}(t) - \bar{y}) \in \mathbb{R}.$$

With y^{kla} being the systems output for the initial state configuration $x_0^{kla} = d_k S_l f_a + \bar{x}$.

Empirical Cross Gramian[8] (WX)

For sets E_u, E_x, R_u, R_x, Q_u, Q_x, input $\bar{u}$ during steady state $\bar{x}$ with output $\bar{y}$, the **empirical cross gramian** is given by:

$$W_X = \frac{1}{|Q_u||R_u|m|Q_x||R_x|} \sum_{h=1}^{|Q_u|} \sum_{i=1}^{|R_u|} \sum_{j=1}^{m} \sum_{k=1}^{|Q_x|} \sum_{l=1}^{|R_x|} \frac{1}{c_h d_k} \int_0^\infty T_l \Psi^{hijkl}(t) T_l^* dt$$

$$\Psi_{ab}^{hijkl}(t) = f_b^* T_k^* \Delta x^{hij}(t) e_i^* S_h^* \Delta y^{kla}(t)$$

$$\Delta x^{hij}(t) = (x^{hij}(t) - \bar{x})$$

$$\Delta y^{kla}(t) = (y^{kla}(t) - \bar{y}).$$

Where x^{hij} and y^{kla} being states and output for the input $u^{hij}(t) = c_h S_i e_j u(t) + \bar{u}$ and initial state $x_0^{kla} = d_k T_l f_a + \bar{x}$ respectively.

[8] for SISO: [Streif'06], for MIMO: [Himpe'13]

Empirical Sensitivity Gramian (WS)

The sensitivity gramian encapsulates the controllability gramian.
$\rightarrow$ Same computation with the empirical controllability gramian.

Empirical Identifiability Gramian (WI)

The identifiability gramian encapsulates the observability gramian.
$\rightarrow$ Same computation with the empirical observability gramian.

Empirical Joint Gramian (WJ)

The joint gramian encapsulates the cross gramian.
$\rightarrow$ Same computation with the empirical cross gramian.

Research Question
(ay, there is the rub)

Hyperbolic Network Model

$$\dot{x}(t) = A_\theta \tanh(K_\theta x(t)) + B_\theta u(t)$$
$$y(t) = C_\theta x(t)$$

State and parameter space dimensions:

$$dim(x) = n$$
$$dim(u) = dim(y) = m$$
$$\Rightarrow dim(\theta) = n^2 + 2mn + n$$

Hyperbolic Network Model

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$$y(t) = C_\theta x(t)$$

State and parameter space dimensions:

$$dim(x) = n$$
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$$\Rightarrow dim(\theta) = n^2 + 2mn + n$$

Bayesian Inverse Problem

$$h(x, \theta) = y + \epsilon$$

- Noise: $\epsilon = N(0, \sigma^2)$
- Bayes Rule: $P(\theta | y_d) = \frac{P(y_d | \theta) P(\theta)}{P(y_d)} \propto P(y_d | \theta) P(\theta)$
- Prior: $P(\theta)$
- Likelihood: $P(y_d | \theta)$
- Evidence: $P(y_d)$
- Posterior: $P(\theta | y_d)$

Robust Reduction

- If the parameters are uncertain
- in example, if they are a Gaussian distribution
- reduction needs to consider this uncertainty

Empirical Gramians to the rescue!

- for W_I and W_J or W_C and W_S
- parameters are or can be included in the perturbed components
- thus parameters uncertainty is taken care of.

Implementation and Numerical Experiments

(almost over, now its mainly pictures)

emgr - Empirical Gramian Framework

Gramians:

- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

Features:

- Uniform Interface
- Compatible with MATLAB & OCTAVE
- Vectorized & Parallelizable
- Open-Source licensed

More info at: `http://gramian.de`

Simplified Hyperbolic Network Model

$$\dot{x}(t) = A_\theta \tanh(x(t)) + Bu(t)$$
$$y(t) = Cx(t)$$

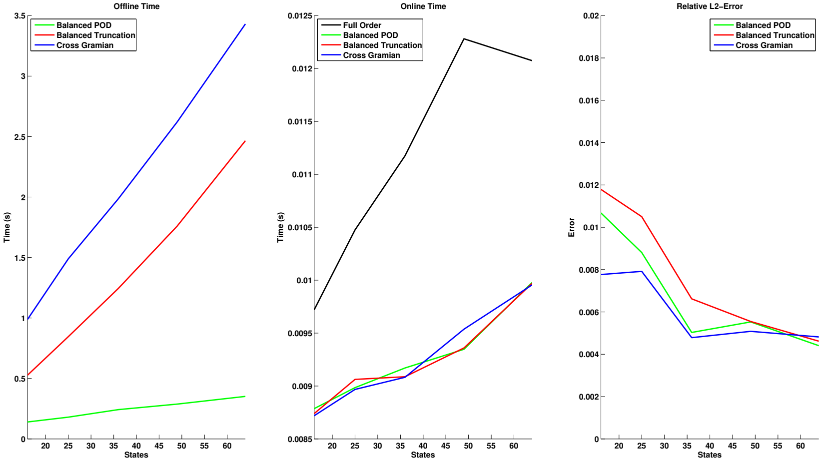
Assumptions:

$$A = A^T$$
$$C = B^T$$
$$P(\theta) = (-\mathbb{1}, 1)$$

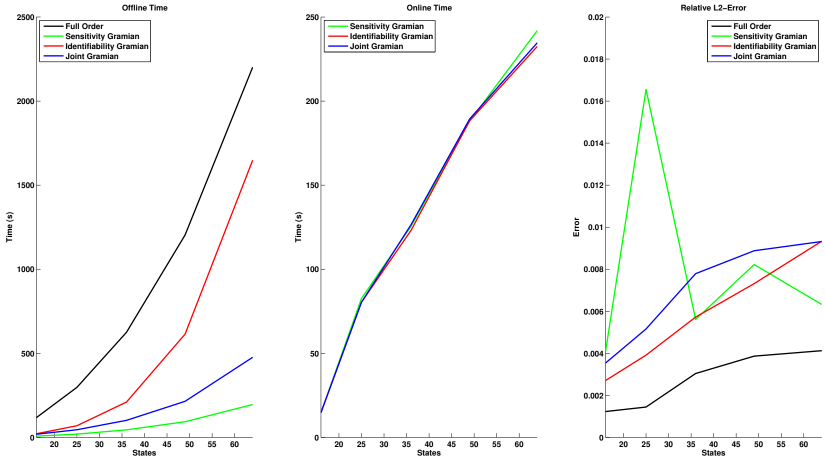
Dimensions:

$$n = \{16, 25, 36, 49, 64\}$$
$$\Rightarrow p = \{256, 625, 1296, 2401, 4096\}$$
$$m = o = \{4, 5, 6, 7, 8\}$$

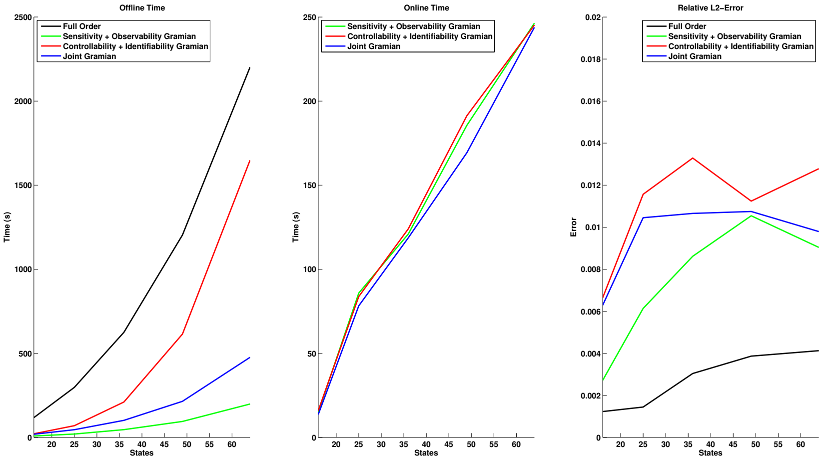
State Reduction (HNM)



Parameter Reduction (HNM)



Combined Reduction (HNM)



FAQ

- Why the Cross-Gramian?
  $\rightarrow$ No balancing required!
- Why Empirical Gramians?
  $\rightarrow$ Allow nonlinear systems, can be data-driven!
- Why the Joint Gramian?
  $\rightarrow$ Best compromise between speed and flexibility!
- Why Bayesian Inversion?
  $\rightarrow$ Prior distribution helps snapshot generation!
- Why Octave?
  $\rightarrow$ Compatible to Matlab and it is Open Source!

tl;dl

- Combined Reduction: reduction of states and parameters.
- Joint Gramian: enables combined reduction.
- Empirical Joint Gramian: nonlinear combined reduction.

- Get the source code: `http://j.mp/hnm13` .

Thanks!

Shameless Plug

Use **emgr**

for:

- Model Order Reduction
  - State Reduction
  - Parameter Reduction
  - Combined Reduction
- System Identification
  - Parameter Identification
  - Sensitivity Analysis
  - Decentralized Control

with:

- First & Second Order
- Linear & Nonlinear
- Parametrized & Parametric Systems

Download, Documentation, Demos:

`http://gramian.de`