Model Reduction for Complex Hyperbolic Networks

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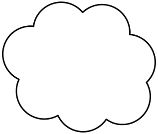
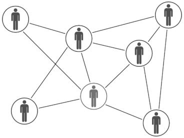
Hyperbolic Networks

- Internet



source: clker.com

- Social Networks



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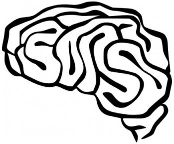
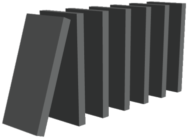
- Brain



source: clker.com

- Causality Networks



Hyperbolic Network Attributes

- Network grows over times
- Each time step a new node is born
- A newborn node connects preferably to older nodes

Side note on ambiguity:

- This is different from a control system with hyperbolic-tangent state transformation!
- Also called "Hyperbolic Network", but a different beast.

Hyperbolic Network Generation [Krioukov et al'12]

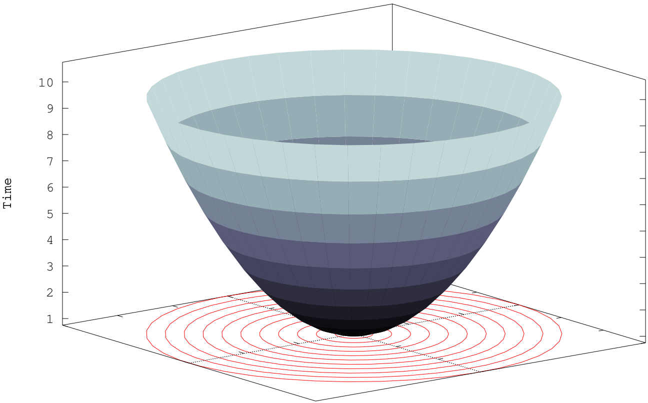
Each time step draw a node from uniform distribution on the circle:

$$x_i = U(\mathbb{S}^1) = (U([0, 2\pi]), r_i = 2 \ln \frac{i}{v})$$

The new node x_i connects to all existing nodes for which:

$$r_j + 2 \ln(\pi - |\pi - |\alpha_i - \alpha_j||) < 2, \ \forall j < i, \ j = 0 \ldots i - 1$$

Space-Time Visualization



Relation to Linear Control Systems

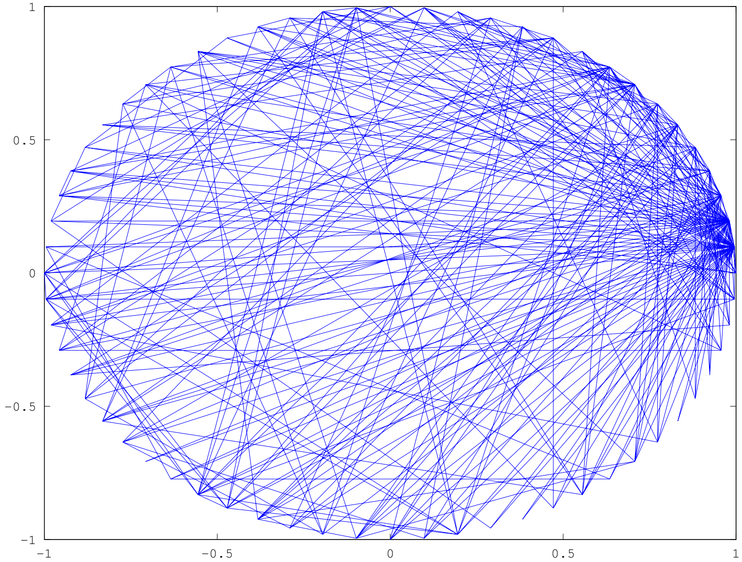
Linear Dynamical System:

$$\dot{x}(t) = A(t)x(t)$$

Time-Varying Parametrized Linear Control System:

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

(Sample) Adjacency Matrix



(Projection-Based) Model Reduction

Large-Scale Control System:

- State: $\dim(x) \gg 1$
- Input: $\dim(u) \ll \dim(x)$
- Output: $\dim(y) \ll \dim(x)$

Projections U, V:

$$\tilde{A} = VAU,\ \tilde{B} = VB,\ \tilde{C} = CU,\ \tilde{x}_0 = Vx_0$$

Reduced Order Model:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$
$$\tilde{y} = \tilde{C}\tilde{x}$$

- $\dim(\tilde{x}) \ll \dim(x)$
- $\|y - \tilde{y}\| \ll 1$

Gramian-Based State Reduction [Moore'81]

Controllability:

Controllability Gramian:

$$W_C := \int_0^\infty e^{At} B B^T e^{A^T t} \mathrm{d}t$$

$$\Rightarrow A W_C + W_C A^T = -B B^T$$

Observability:

Observability Gramian:

$$W_O := \int_0^\infty e^{A^T t} C^T C e^{At} \mathrm{d}t$$

$$\Rightarrow W_O A^T + A W_O = -C^T C$$

$\rightarrow$ Balanced Truncation

Cross Gramian [Fernando & Nicholson'83]

Cross Gramian:

$$W_X := \int_0^\infty e^{At} B C e^{At} \mathrm{d}t$$
$$\overset{1.}{\Rightarrow} A W_X + W_X A = -BC$$
$$\overset{2.}{\Rightarrow} W_X^2 = W_C W_O$$

$\rightarrow$ Approximate Balancing

System must be:

1. square: $\#$Inputs $=$ $\#$Outputs
2. symmetric: $Ce^A B = (Ce^A B)^T$

Non-Symmetric Cross Gramian [Sorensen & Antoulas'02]

Embedding into a Symmetric System:

$$J \in \mathbb{R}^{N \times N},\ AJ = JA^T$$
$$\dot{x} = Ax + \begin{pmatrix} JC^T & B \end{pmatrix} u'$$
$$y' = \begin{pmatrix} C \\ B^T J^{-1} \end{pmatrix} x$$

Trivial Symmetrizer:

$$A = A^T \Rightarrow J = \mathbb{1}$$
$$\dot{x} = Ax + \begin{pmatrix} C^T & B \end{pmatrix} u'$$
$$y' = \begin{pmatrix} C \\ B^T \end{pmatrix}$$

Parameter Identification [Geffen'08, H.'14]

Parameter Augmented System:

$$\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

Joint Gramian (Cross Gramian of Parameter Augmented System):

$$W_J = \begin{pmatrix} W_X & W_M \\ 0 & 0 \end{pmatrix}$$

Cross-Identifiability Gramian (Schur Complement of Symmetric Part of Joint Gramian):

$$W_{\ddot{I}} := -\frac{1}{2} W_M (W_X + W_X^T)^{-1} W_M^T$$

Combined Reduction [H.'14]

Parameter Reduction:

$$W_I \overset{SVD}{=} UDV^T$$
$$\rightarrow U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

Reduced Parameters:

$$\tilde{\theta} = U_1 \theta$$

State Reduction:

$$W_X \overset{SVD}{=} UDV^T$$
$$\rightarrow U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

State-Reduced System:

$$\tilde{A} = U_1^T A U_1$$
$$\tilde{B} = U_1^T B$$
$$\tilde{C} = C U_1$$
$$\tilde{x}_0 = U_1^T x_0$$

Empirical Gramians

System Gramians:

$$W_C = \int_0^\infty (e^{At} B)\,(e^{At} B)^T \mathrm{d}t$$

$$W_O = \int_0^\infty (e^{A^T t} C^T)\,(e^{A^T t} C^T)^T \mathrm{d}t$$

$$W_X = \int_0^\infty (e^{At} B)\,(e^{A^T t} C^T)^T \mathrm{d}t$$

Integrands are just impulse responses!

$\rightarrow$ Compute gramians empirically.

(Empirical gramians can also be computed for nonlinear systems!)

Empirical Gramians [Lall et al'99]

Empirical Controllability Gramian

$$W_C = \int_0^\infty (e^{At}B)\,(e^{At}B)^T \mathrm{d}t$$

$$W_C'(u_\delta) = \int_0^\infty x_{u_\delta}(t)\, x_{u_\delta}^T(t)\mathrm{d}t$$

$$W_C'' = \frac{1}{|U_\delta|}\sum_{u\in U_\delta} W_C'(u)$$

Empirical Observability Gramian

$$W_O = \int_0^\infty (e^{A^T t}C^T)\,(e^{A^T t}C^T)^T \mathrm{d}t$$

$$W_{O,(i,j)}'(x_0) = \int_0^\infty y_{x_0\circ e_i}^T(t)\, y_{x_0\circ e_j}(t)\mathrm{d}t$$

$$W_O'' = \frac{1}{|X_0|}\sum_{x\in X_0} W_O'(x)$$

Empirical Cross Gramian [Streif et al'06, Streif'09, H.'14]

Empirical Cross Gramian

$$W_X = \int_0^\infty (e^{At} B)\,(e^{A^T t} C^T)^T \mathrm{d}t$$

$$W'_{X,(i,j)}(u_\delta, x_0) = \int_0^\infty e_i^T x_{u_\delta}(t)\; f_u^T y_{e_i \circ x_0}(t) \mathrm{d}t$$

$$W''_X = \frac{1}{|U_\delta \times X_0|} \sum_{u \in U_\delta, x \in X_0} W'_x(u, x)$$

Uncertainty Quantification

Uncertainty Quantification and Empirical Gramians

- The perturbation sets for the empirical gramians are very flexible
- Uncertainties can be incorporated into perturbation sets ($U_\delta,\ X_0$)
- Thus uncertainties can be included with regards to:
  - Inputs
  - Initial States
  - Parameters (!)

Numerical Experiments

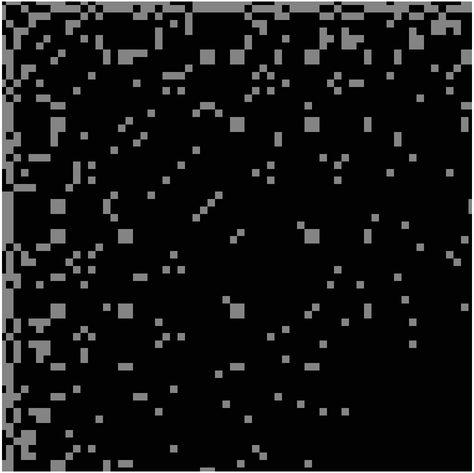
Sample Network:

- 64 Nodes
- 8 Inputs and Outputs
- 2016 Parameters
- $A = A^T$

Symmetric Parametrized Linear Time-Varying Control System:

$$\dot{x} = A(\theta(t))x + \begin{pmatrix} C^T & B \end{pmatrix} u'$$
$$y' = \begin{pmatrix} C \\ B^T \end{pmatrix} x$$

System Matrix



emgr - Empirical Gramian Framework

Gramians:

- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Cross Gramian
- Empirical Linear Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

Features:

- Uniform Interface
- Compatible with MATLAB & OCTAVE (& FREEMAT)
- Vectorized & Parallelizable
- Open-Source licensed

More info at: `http://gramian.de`

Numerical Results (Singular Values)

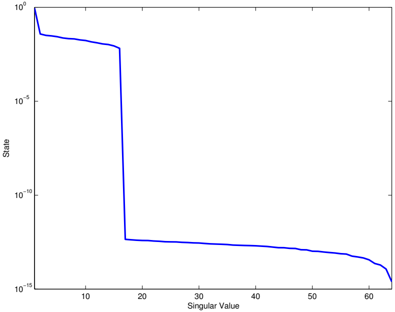


Figure: Cross Gramian Singular Values

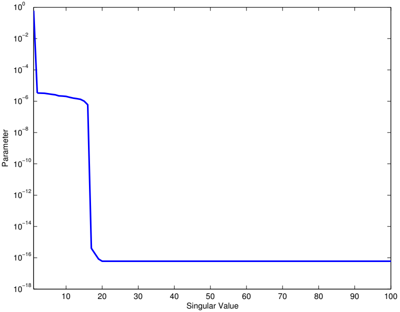


Figure: First 100 Cross-Identifiability Gramian Singular Values

Numerical Results (Impulse Response)

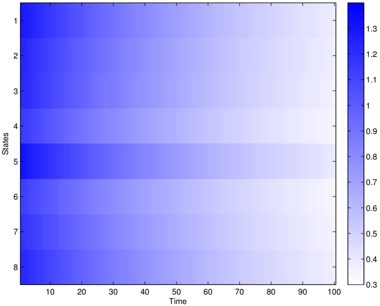


Figure: Full-Order Model Impulse Response

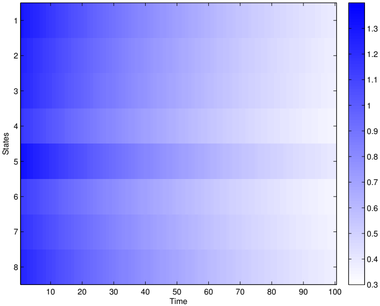
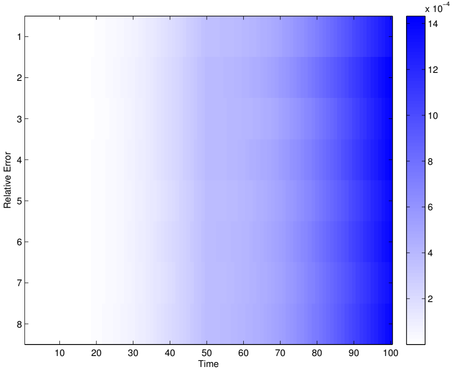


Figure: Reduced Order Model Impulse Response

Numerical Results (Comparison)



Original Time [s]:	0.0216
Offline Time [s]:	128.2395
Online Time [s]:	0.0143
Relative L^2-Error:	0.00034
Memory Reduction:	85%

Related Work

We are using Empirical Gramians also for:

- Combined Reduction for Connectivity Analysis of fMRI/fNIRS, EEG/MEG data
- Parameter Identification for EEG/MEG models
- Model Order Reduction for Dynamic Causal Modeling

Approach:

- C. Himpe and M. Ohlberger, "Cross-Gramian Based Combined State and Parameter Reduction for Large-Scale Control System", Mathematical Problems in Engineering (Accepted), 2014.

tl;dl

- Empirical Cross Gramian
- Empirical Joint Gramian (Cross-Identifiability Gramian)
- **Non-Symmetric** Parametrized Time-Varying Linear Control System

- Preprint: `http://arxiv.org/pdf/1310.0761`
- Source Code: `http://j.mp/ecc14_code`
- emgr 2.2: `http://gramian.de`

Thanks!