Westfälische Wilhelms-Universität Münster

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Combined State and Parameter Reduction

(for Input-Output Systems)

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Disclaimer

1. There will be no flashy images.
2. There will be formulas.

Why Model Reduction?

My simulation is based on a model, ...

- ... that is large.
- ... that needs to be simulated many times.
- ... that has to be simulated in n seconds.

What it usually boils down to:
"It takes too long!"

Popular Beliefs

1. “I just buy a faster computer.”
   - Moore’s Law
   - The memory bottleneck
   - Resource coverage (DM, NUMA, SMP, SMT, SIMD, GPGPU)
2. “I just use a coarser grid.”
   - Detail Resolution
   - Numerical Properties
   - Information Disregard

A Mathematical Model

(Pretty) General Input-Output System:

$$\dot{x}(t) = f(x(t), u(t), \theta),$$
$$y(t) = g(x(t), u(t), \theta),$$
$$x(0) = x_0$$

System Components:

- $x(t)$ - State
- $u(t)$ - Input / Control
- $y(t)$ - Output
- θ - Parameter

One of Many Interpretations

- x_0 is a equilibrium state of the system.
- $u(t)$ is an external perturbation
- to a system with dynamic behavor $x(t)$.
- $y(t)$ is a measurement from a few sensors.

The Poster Child

Linear (Time-Invariant) System:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$
$$x(0) = x_0$$

In The Wild

Input-Output Systems are used in:

- Industrial Control
- Mechanics
- Electro Dynamics
- Fluid Dynamics
- Reaction Networks
- Neuro Imaging
- Network Dynamics
- ...

Play It Again, Sam

Many-Query Settings:

- Optimal Control
- Model Predictive Control
- Model Constraint Optimization
- Inverse Problems
- Sensitivity Analysis
- »Uncertainty Quantification«

Setting the Stage

Now, what means large?

- High-Dimensional State-Space: $\dim(x(t)) \gg 1$
- High-Dimensional Parameter-Space: $\dim(\theta) \gg 1$

(Input- and output-space are usually small.)

Enter MOR

Model Order Reduction (MOR):

- Low-Dimensional State-Space: $\dim(x_r(t)) \ll \dim(x(t))$
- Low-Dimensional Parameter-Space: $\dim(\theta_r) \ll \dim(\theta)$
- Model Reduction Error[1] $\|y - y_r\| \ll 1$ (!)

[1] In a suitable norm.

Intermission I

What we use model order reduction (MOR) for:

Network Connectivity Reconstruction

(from neuroimaging data for brain connectivity analysis)

Intermission II

We have:

- Low-dimensional time-series measurements
- from a large network of known size,
- which is controllably perturbed.

We want:

- statistics
- on the inter-node connectivity

(This is a bayesian inverse problem treated with model constraint optimization)

It's A Bird ... It's A Plane ... It's A

Reduced Order Model (ROM):

$$\dot{x}_r(t) = f_r(x_r(t), u(t), \theta_r),$$
$$y_r(t) = g_r(x_r(t), u(t), \theta_r),$$
$$x_r(0) = x_{r,0}$$

Project Me If You Can

Projection-Based ROM[2]:

$$
\begin{aligned}
\dot{x}_r(t) &= Vf(Ux_r(t), u(t), \Pi\theta_r), \\
y_r(t) &= g(Ux_r(t), u(t), \Pi\theta_r), \\
x_r(0) &= Vx_0, \\
\theta_r &= \Lambda\theta
\end{aligned}
$$

With:

- (Low-rank) state-space projection $\{U, V\}$
- (Low-rank) parameter projection $\{\Pi, \Lambda\}$

[2] We delibaretly ignore the lifting bottleneck here.

A New State-Space Hope

Recipe for a reducing state-space projection:

1. Select a criteria for importance of states.
2. Transform the sytem so states are sorted.
3. Discard the least important states.

(We choose input-output energy transfer.)

The Dual Duo

Controllability:

$$\mathcal{C}(u) := \int_{-\infty}^{0} \mathrm{e}^{-At} Bu(t)\mathrm{d}t$$

(How well can the states be driven by input)

Observability:

$$\mathcal{O}(x_0)(t) = C\,\mathrm{e}^{At}\,x_0$$

(How well changes in the state are reflected by the output)

Back To The Future

System Gramians:

$$W_C := \mathcal{C}\mathcal{C}^*$$
$$W_O := \mathcal{O}^*\mathcal{O}$$

Relation to the Hankel Operator $H := \mathcal{O}\mathcal{C}$:

$$\begin{aligned}
\sqrt{\lambda(W_C W_O)} &= \sqrt{\lambda(\mathcal{C}\mathcal{C}^*\mathcal{O}^*\mathcal{O})} \\
&= \sqrt{\lambda(\mathcal{C}^*\mathcal{O}^*\mathcal{C}\mathcal{O})} \\
&= \sqrt{\lambda((\mathcal{O}\mathcal{C})^*\mathcal{O}\mathcal{C})} \\
&= \sqrt{\lambda(H^* H)} \\
&= \sigma(H)
\end{aligned}$$

(H maps past inputs to future outputs and has finite rank.)

Weighing Yin and Yang

Balancing:

$$W_C^{\frac{1}{2}} W_O^{\frac{1}{2}} \overset{SVD}{=} UDV$$

(U and V constitute a balancing transformation)

Truncating:

$$U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}, \qquad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

(Partitioning is based on the decay of the σ_i)

Balanced Truncation:
"If I can't control it or observe it I don't need it."

Symmetric Encounter ...

Hankel Operator:

$$H = \mathcal{O}\mathcal{C}$$

What if H is symmetric?

$$\begin{aligned} H &= H^* \\ \Rightarrow \mathcal{O}\mathcal{C} &= (\mathcal{O}\mathcal{C})^* \\ \Rightarrow \mathcal{C}\mathcal{C}^*\mathcal{O}^*\mathcal{O} &= \mathcal{C}(\mathcal{O}\mathcal{C})^*\mathcal{O} \\ &= \mathcal{C}(\mathcal{O}\mathcal{C})\mathcal{O} \\ &= (\mathcal{C}\mathcal{O})(\mathcal{C}\mathcal{O}) \end{aligned}$$

… Of The Third Kind

A third system gramian - the cross gramian:

$$W_X := \mathcal{C}\mathcal{O}$$

(Controllability and observability in one matrix!)

Approximate Balancing:

$$W_X \overset{SVD}{=} UDV$$

Direct Truncation:

$$U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}, \qquad V_1 = U_1^T$$

By Empirical Decree

How to compute these system gramians?

- Solving matrix equations
- Use empirical gramians $(*)$

The Parameter-Space Strikes Back

Same recipe:

1. Select critera
2. Sort states
3. Discard tail

(Spoiler alert: We will use state-to-ouput influence)

A Parameter In A Tuxedo ...

Parameter Augmented System:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} f(x(t), u(t), \theta) \\ 0 \end{pmatrix},$$

$$y(t) = g(x(t), u(t), \theta),$$

$$\begin{pmatrix} x(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ \theta \end{pmatrix}$$

Double Cross

Block structure of the joint gramian:

$$W_J = \begin{pmatrix} W_X & W_M \\ 0 & 0 \end{pmatrix}$$

(The joint gramian is the cross gramian of an augmented system.)

Cross-Identifiability gramian:

$$W_{\ddot{I}} = 0 - \frac{1}{2} W_M^T (W_X + W_X^T)^{-1} W_M$$

($W_{\ddot{I}}$ encodes the "observability" of parameters.)

Parameter Truncation:

$$W_{\ddot{I}} \overset{SVD}{=} \Pi \Delta \Lambda \rightarrow \Pi = \begin{pmatrix} \Pi_1 & \Pi_2 \end{pmatrix}$$

Return Of The Combined Reduction

State-space projection:

$$W_X \overset{SVD}{=} UDV$$

Parameter-space projection:

$$W_{\ddot{I}} \overset{SVD}{=} \Pi\Delta\Lambda$$

Combined state and parameter ROM:

$$\begin{aligned}
\dot{x}_r(t) &= U_1^T f(U_1 x_r(t), u(t), \Pi_1\theta_r), \\
y_r(t) &= g(U_1 x_r(t), u(t), \Pi_1\theta_r), \\
x_r(0) &= U_1^T x_0, \\
\theta_r &= \Pi_1^T \theta
\end{aligned}$$

Not Too Nonlinear

Hyperbolic Network Model:

$$
\begin{aligned}
\dot{x}(t) &= A \tanh(K(\theta)x(t)) + Bu(t), \\
y(t) &= Cx(t), \\
x(0) &= x_0
\end{aligned}
$$

Better Call `emgr`

`emgr` - Empirical Gramian Framework (Version: 3.6, 10/2015)

Empirical Gramians:

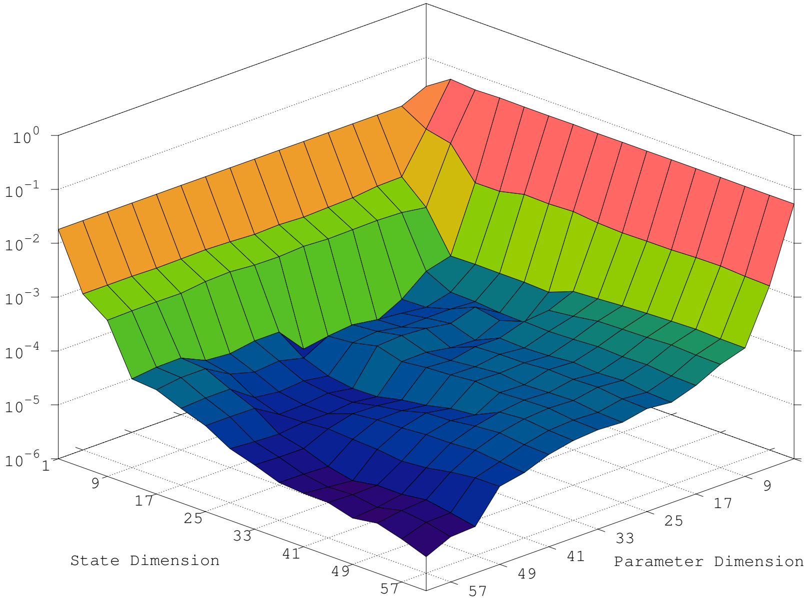
- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Linear Cross Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramians

Features:

- Custom Solver Interface
- Non-Symmetric Cross Gramian
- Compatible with OCTAVE and MATLAB
- Vectorized and Parallelizable
- Permissive Open Source License (BSD 2-Clause)

More info at: `http://gramian.de`

$L_2 \otimes L_2$ output error for varying reduced state and parameter dimensions



tl;dl

Summary:

- Combined state and parameter reduction
- using empirical gramians
- for nonlinear input-output systems.

`wwwmath.uni-muenster.de/u/himpe`

Thanks!