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# The Versatile Cross Gramian

## for System-Theoretic Model Reduction and More

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# System of Interest

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), \theta) \\ y(t) &= g(x(t), u(t), \theta)\end{aligned}$$

(A parametrized input-output system)

# Controllability & Observability

$$\mathcal{C} : L_2 \rightarrow \mathbb{R}^N$$

$$\mathcal{O} : \mathbb{R}^N \rightarrow L_2$$

(These dual operators are essential to system theory)

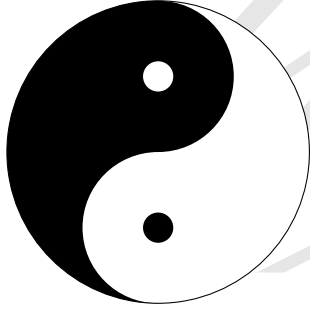
# The Cross Gramian (Matrix)

$$W_X := \mathcal{C} \circ \mathcal{O}$$

(aka  $W_{CO}$ , introduced by [FERNANDO & NICHOLSON'83] )

# Balancedness

Controllability



Observability

# One System Gramian To Rule Them All!


For:

- Model Reduction
- Sensitivity Analysis
- System Identification
- Decentralized Control
- Parameter Identification
- **Combined State and Parameter Reduction**

Of:

- Symmetric Systems
- Orthogonally Symmetric Systems
- Gradient Systems
- Non-Symmetric Systems
- Non-Square Systems
- Nonlinear Systems


# Do You Want To Know More?



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## The Versatile Cross Gramian for System Theoretic Model Reduction and More

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**Abstract:**  
For input-output systems, the cross gramian matrix encodes controllability and observability information into a single matrix, which are essential to system-theoretic applications. This systems gramian can be used, for example, for model order reduction, sensitivity analysis, system identification, decentralized control and parameter identification. Beyond linear systems, the cross gramian is also available for parametric, non-symmetric, non-square and nonlinear systems.

**Cover Systems**  
 $X(t) = A(t)X(t) + B(t)u(t)$   
 $y(t) = C(t)X(t) + D(t)u(t)$

**Linear Systems**  
 $X(t) = AX(t) + Bu(t)$   
 $y(t) = Cx(t) + Du(t)$

**Definition:**  
The cross gramian matrix  $W_2$  is defined as composition of the controllability operator  $C$  with the observability operator  $O$ :

$$W_2 = CO^* = \int_0^{\infty} e^{A^*t} B C^* e^{At} dt$$

**Cross Gramian**

Model Reduction

Sensitivity Analysis

System Identification

Decentralized Control

Non-Symmetric Systems

Parameter Identification

**Classic Computation:** The cross gramian of a square linear system can be computed as the solution to a Sylvester matrix equation:

$$AW_2 + W_2A = -BC$$

**Empirical Computation:** The cross gramian of a general control system can be computed as product of state and adjoint or output impulse responses:

$$W_2 = XY^{*T,2,2}$$

**Symmetric Systems:** A system is symmetric if its transfer function  $G(s)$  is symmetric:  $G(s) = G(s)^*$

**Hermitian Operator:** The matrix operator  $H$  is defined as the composition of  $C^*$  with  $C$ :  $H = CC^*$

**State-Space Symmetric:** An input-output symmetric system is called symmetric as system:  $W_2 = W_2^* = W_2$

**Decentralized Control:** Decomposition of a MIMO system into a set of SISO systems retaining the structural input-output relations.

**Non-Symmetric Systems:** For a square non-symmetric system  $W_2 = \int_0^{\infty} e^{A^*t} B C^* e^{At} dt = \int_0^{\infty} e^{A^*t} B C^* e^{At} dt$

**Generalized Symmetry:** System symmetry can be defined by general and first-order linear systems by:  $\text{ker}(C) = \text{ker}(C^*)$

**Augmented Systems:** Treating uncertainty as state with an augmented system:  $\begin{pmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{pmatrix} = \begin{pmatrix} A & B \\ F & G \end{pmatrix} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$

**Core Property:** For symmetric systems, the following relation between the cross gramian holds:  $W_2 = W_2^* = W_2$

**Trace:** The trace of the Hermitian operator  $H$  is equal to the trace of the cross gramian:  $\text{tr}(W_2) = \text{tr}(H)$

**Minimality:** If the cross gramian has full rank, then the system is controllable, observable and thus of minimal order.

**Eigen Problem:** For a square MIMO system,  $W_2$  can be written as each of SISO subsystem cross gramians:  $W_2 = \sum_{i=1}^n e^{A_i^*t} B_i C_i^* e^{A_i t} dt$

**Orthogonal Symmetry:** The cross gramian inherits its orthogonality symmetric systems:  $W_2 = A_i^{-*} e^{A_i^* t} B_i C_i^* e^{A_i t}$

**Gradient System:** A gradient system is symmetric and can be written as gradient and first-order linear systems by:  $\dot{X}(t) = -\nabla V(X(t))$

**Joint Gramian:** The joint gramian is the cross gramian of an augmented system:  $W_2 = \begin{pmatrix} W_2 & 0 \\ 0 & 0 \end{pmatrix}$

**Hermitian Singular Values:** The Hermitian singular value decomposition of the cross gramian:  $W_2 = U \Sigma U^*$

**System Gain:** The system gain equals the trace of the transfer function of the system:  $\text{tr}(W_2) = \text{tr}(C(A^{-1}B))$

**Cauchy Index:** The signum of the cross gramian matrix is equal to the system's Cauchy index:  $\text{sgn}(W_2) = \text{tr}(C(s))$

**Integrable Observers:** The relative trace output difference can be measured for systems by the subsystem gain:  $W_2 = W_2^* = W_2$

**Approximate Cross Gramian:** An approximate cross gramian is given by substituting using system  $A$ :  $W_2 \approx \int_0^{\infty} e^{A^*t} B C^* e^{At} dt$

**Generalized Cross Gramian:** The general cross operator is given by substituting using system  $A$ :  $\text{ker}(W_2) = \text{ker}(C)$

**Controllability:** The system is controllable if  $W_2$  is nonsingular based observability information:  $W_2 = W_2^* = W_2^{-1} W_2$

**Model Reduction:** An approximate balancing operation is given by the left singular vectors of a TDS of  $W_2$ :  $U = U_1 + \dots + U_n$

**Stability Analysis:** Testing parameters in terms the gain can be used stability measure:  $\|W_2\| = \|C(A^{-1}B)\|$

**Singularity Index:** The rank of the controllability operator of the actual system  $(W_2, B, C)$  equals the singularity index.

**Participation Matrix:** A matrix of all input-output combinations of all subsystems means the balancing measure yields the balancing measure  $W_2$ .

**Non-Symmetric Cross Gramian:** The non-symmetric cross gramian is defined as each of all subsystem cross gramians:  $W_2 = \sum_{i=1}^n e^{A_i^*t} B_i C_i^* e^{A_i t} dt$

**Symmetry Test:** A nonlinear system is symmetric if its cross gramian is symmetric and  $C(s) = C^*(s)$

**Modification:** An SVD of the cross gramian yields the dominant subsystem:  $W_2 = U \Sigma U^*$

**Read Me:**  
 C. Hempel and M. Ohlberger "A note on the non-symmetric cross gramian" *arXiv Preprint*, math/0512083(2005-12-14), 2005.  
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 L.A. Hernandez and F.H. Schuler "Analysis of stability of transfer function reduction of control systems" *Automatica and Systems Control*, 19(2):100-108, 1989.  
 T.C. Lennox, K. Fujimoto, and J.M.A. Schlegel "Transfer value analysis of non-linear systems" *Transactions on Automatic Control of the IEEE*, 19(7):1079-1086, 1974.

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