Westfälische Wilhelms-Universität Münster

Applied Mathematics Münster

Model Reduction for Parameterized Systems

Smaller Better Faster Stronger

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Motivation

Scope:

- Large-scale dynamical systems,
- or discretized partial differential equations
- together with an observer
  (i.e. sensors generating measurements)

Many-Query Settings:

- Model-constraint Optimization / Inverse Problems
- Optimal Control / Model Predictive Control
- Heuristic Development

Control Systems

Linear State-Space System:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + F, \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0 \end{aligned}$$

General State-Space System:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= g(x(t), u(t)), \\ x(0) &= x_0 \end{aligned}$$

System Dimension (Order): $N := \dim(x(t))$

Aim

- Enable simulations of high-dimensional systems.
- Accelerate simulations in many-query settings.
- Identify driving components of input-output systems.

Outline

1. Model Reduction
2. Gramian-Based Model Reduction
3. Empirical Gramians
4. Parametric Model Reduction
5. Numerical Experiments

Model Order Reduction (MOR)

Typically:

- $\dim(x(t)) \gg 1$
- $\dim(u(t)) \ll \dim(x(t))$
- $\dim(y(t)) \ll \dim(x(t))$

Want:

$$\begin{aligned} \dot{x}_r(t) &= f_r(x_r(t), u(t)), \\ y_r(t) &= g_r(x_r(t), u(t)), \\ x_r(0) &= x_{r,0} \end{aligned}$$

Such that:

- $\dim(x_r(t)) \ll \dim(x(t))$
- $\|y - y_r\| \ll 1$

Reduced Order Model (ROM)

General Case:

$$\begin{aligned}\dot{x}_r(t) &= f_r(x_r(t), u(t)),\\ y_r(t) &= g_r(x_r(t), u(t)),\\ x_r(0) &= x_{r,0}\end{aligned}$$

Linear Case:

$$\begin{aligned}\dot{x}_r(t) &= A_r x_r(t) + B_r u(t),\\ y_r(t) &= C_r x_r(t),\\ x_r(0) &= x_{r,0}\end{aligned}$$

Projection-Based Model Reduction

Petrov-Galerkin Projection (Two-Sided):

$$U \in \mathbb{R}^{N\times n}, V \in \mathbb{R}^{n\times N}, n \ll N, VU = \mathbb{1}$$

Galerkin Projection (One-Sided):

$$U \in \mathbb{R}^{N\times n}, V = U^T, n \ll N, U^T U = \mathbb{1}$$

General Case:

$$\dot{x}_r(t) = Vf(Ux_r(t), u(t)),$$
$$y_r(t) = g(Ux_r(t), u(t)),$$
$$x_r(0) = Vx_0$$

Linear Case:

$$\dot{x}_r(t) = VAUx_r(t) + VBu(t),$$
$$y_r(t) = CUx_r(t),$$
$$x_r(0) = Vx_0$$

Hankel Singular Values (HSV)

Convolution Operator S (Infinite Rank):

$$y(t) = S(u)(t) = \int_0^\infty Ce^{A(t-\tau)}Bu(\tau)d\tau$$

Time-Flip Operator F:

$$F(u)(t) = u(-t)$$

Hankel Operator H (Finite Rank):

$$H(u)(t) = (S \circ F)(u)(t) = \int_{-\infty}^0 Ce^{A(t-\tau)}Bu(\tau)d\tau$$

- maps past inputs to future outputs: $u \overset{\mathcal{C}}{\mapsto} \mathbb{R}^N \overset{\mathcal{O}}{\mapsto} y' \Rightarrow H = \mathcal{O}\mathcal{C}$
- singular values of the Hankel operator σ_i are system invariants,
- describing the input-output coherence of the states.

Controllability & Observability (System Gramians)

Controllability Gramian:

$$W_C := \mathcal{C}\mathcal{C}^T = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$$

Lyapunov Equation:

$$AW_C + W_CA^T = BB^T$$

Observability Gramian:

$$W_O := \mathcal{O}^T\mathcal{O} = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$$

Lyapunov Equation:

$$A^TW_O + W_OA = C^TC$$

Balanced Truncation

System Gramian Relation to HSVs:

$$\sigma_i = \sqrt{\lambda_i(W_C W_O)}$$

Balancing Transformation[1]:

$$\sqrt{W_C}\sqrt{W_O} \stackrel{SVD}{=} UDV$$

Truncated State-Space Petrov-Galerkin Projection U_1, V_1:

$$U = \left(U_1, U_2\right), V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

[1] The matrix square root can be computed i.e. via a Cholesky decomposition $L^T L$ or an SVD $UD^{\frac{1}{2}}V$

Cross Gramian

Cross Gramian (for square systems only):

$$W_X := \mathcal{C}\mathcal{O} = \int_0^\infty e^{At} B C e^{At} dt$$

Sylvester Equation:

$$A W_X + W_X A = BC$$

Direct Truncation

For Symmetric Systems:

$$\sigma_i = |\lambda_i(W_X)|$$

Approximate Balancing Transformation U:

$$W_X \overset{SVD}{=} UDV$$

Truncated State-Reducing Galerkin Projection U_1:

$$U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}$$

Empirical Gramians

System Gramians:

$$W_C = \int_0^\infty (e^{At}B)(e^{At}B)^T dt,$$
$$W_O = \int_0^\infty (Ce^{At})^T(Ce^{At}) dt,$$
$$W_X = \int_0^\infty (e^{At}B)(Ce^{At}) dt$$

- Computable empirically by impulse responses,
- thus also for nonlinear systems;
- initially used in [Moore'81].

Empirical Controllability Gramian [Lall et al.'99]

(Impulse) Input Perturbations:

$$Q_U = \{q_k\}_{k=1\dots K}$$

Sampled State Trajectories:

$$x_{q_k}(t) = \mathcal{C}(u_{q_k})(t) = (\mathcal{C} \circ q_k)(\delta)(t)$$

Empirical Controllability Gramian[2]:

$$\widehat{W}_C = \sum_{k=1}^{K} \int_0^\infty \Psi_C^{q_k}(t)dt$$
$$\Psi_C^{q_k}(t) = x_{q_k}(t) x_{q_k}^T(t)$$

[2] [Lall et al.'99] showed that for linear systems $\widehat{W}_{\mathbf{C}} = W_{\mathbf{C}}$.

Empirical Observability Gramian [Lall et al.'99]

Initial State Perturbations:

$$Q_X = \{p_l\}_{l=1\dots L}$$

Sampled Output Trajectories:

$$y_{p_l}(t) = \mathcal{O}(x_{p_l,0})(t) = (\mathcal{O} \circ p_l)(x_0)(t)$$

Empirical Observability Gramian[3]:

$$\widehat{W}_O = \sum_{l=1}^{L} \int_0^\infty p_l \Psi_O^{p_l}(t) p_l^T \, dt$$

$$\Psi_{O,ij}^{p_l}(t) = y_{p_l,i}(t) y_{p_l,j}(t)$$

[3] [Lall et al.'99] showed that for linear systems $\widehat{W}_{\boldsymbol{O}} = W_{\boldsymbol{O}}$.

Empirical Cross Gramian [Streif et al.'06], [H. & Ohlberger'14]

Input and State Perturbations:

$$Q_U = \{q_k\}_{k=1\dots K}$$
$$Q_X = \{p_l\}_{l=1\dots L}$$

Sampled State and Output Trajectories:

$$x_{q_k}(t) = \mathcal{C}(u_{q_k})(t) = (\mathcal{C} \circ q_k)(\delta)(t)$$
$$y_{p_l}(t) = \mathcal{O}(x_{p_l,0})(t) = (\mathcal{O} \circ p_l)(x_0)(t)$$

Empirical Cross Gramian[4]:

$$\widehat{W}_X = \sum_{k=q}^{K} \sum_{l=1}^{L} \int_0^\infty p_l \Psi_X^{q_k,p_l}(t) p_l^T \, dt$$
$$\Psi_{X,ij}^{q_k,p_l}(t) = p_l x_{q_k,i}(t) q_k y_{p_l,j}(t)$$

[4] [H. & Ohlberger'14] showed that for linear systems $\widehat{W}_{\boldsymbol{X}} = W_{\boldsymbol{X}}$.

Parameterized Systems

Parameterized Systems:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), \theta),\\ y(t) &= g(x(t), u(t), \theta),\\ x(0) &= x_0\end{aligned}$$

Example:

$$\begin{aligned}\dot{x}(t) &= A(\theta)x(t) + Bu(t),\\ y(t) &= Cx(t),\\ x(0) &= x_0\end{aligned}$$

Parametric Model Order Reduction (pMOR)

Want:

$$\begin{aligned}
\dot{x}_r(t) &= f_r(x_r(t), u(t), \theta), \\
y_r(t) &= g_r(x_r(t), u(t), \theta), \\
x_r(0) &= x_{r,0}
\end{aligned}$$

Such that:

- $\dim(x_r(t)) \ll \dim(x(t))$
- $\|y(\theta) - y_r(\theta)\| \ll 1$

Projection-Based Parametric Model Order Reduction:

$$\begin{aligned}
\dot{x}_r(t) &= Vf(Ux(t), u(t), \theta), \\
y_r(t) &= g(Ux(t), u(t), \theta), \\
x_r(0) &= Vx_0
\end{aligned}$$

Empirical Gramians for pMOR

Controllability-Based[5]:

$$\overline{W}_C = \sum_{\theta \in \Theta_{\boldsymbol{h}}} W_C(\theta),$$

Observability-Based[5]:

$$\overline{W}_O = \sum_{\theta \in \Theta_{\boldsymbol{h}}} W_O(\theta),$$

Cross-Gramian-Based[5]:

$$\overline{W}_X = \sum_{\theta \in \Theta_{\boldsymbol{h}}} W_X(\theta).$$

[5] For a discretized parameter space $\Theta_{\boldsymbol{h}}$

Application

Adjacency Matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix}$$

Linear Network Model:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \\ x(0) &= x_0 \end{aligned}$$

Network Dynamics from Output Measurements y_d:

$$\theta_d = \text{argmin}_\theta \, \|y_d - y(\theta)\|_2^2$$

Test Model

Hyperbolic Network Model [Quan et al.'01]:

$$\begin{aligned}\dot{x}(t) &= A\tanh(K(\theta)x(t)) + Bu(t),\\ y(t) &= Cx(t),\\ x(0) &= x_0\end{aligned}$$

System Properties:

- State $x(t) \in \mathbb{R}^{256}$
- Input $u \in L_2^1$
- Output $y \in L_2^1$
- Parameter $\theta \in \mathbb{R}^{256}$ $K(\theta) = \mathrm{diag}(\theta)$

- System Matrix $A \in \mathbb{R}^{256\times 256}$
- Input Matrix $B \in \mathbb{R}^{256\times 1}$
- Output Matrix $C \in \mathbb{R}^{1\times 256}$
- Activation Matrix $K \in \mathbb{R}^{256\times 256}$ (diagonal)

Numerical Results

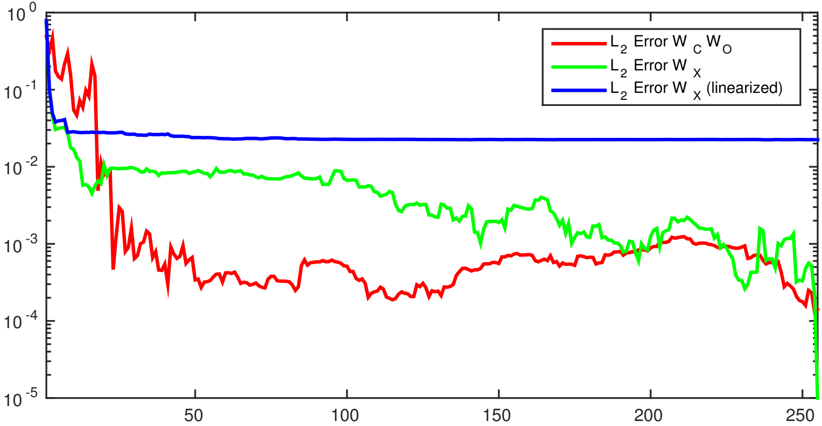

Figure : L_2 output error of the hyperbolic network model for varying state dimension given a uniformly random parameter.

tl;dl

- Parametrized Model Order Reduction
- using empirical gramians
- for linear and nonlinear systems.

- Get the companion code: `http://j.mp/twente15`
- Empirical Gramian Framework: `http://gramian.de`
- Me: `http://wwwmath.uni-muenster.de/u/himpe`

Thanks!

Non-Symmetric Cross Gramian (Bonus)

Linear System Gramian Decentralization:

$$B = \begin{pmatrix} b_1 & \dots & b_J \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & \dots & c_K \end{pmatrix}^T$$

$$\Rightarrow W_C = \sum_{j=1}^{J} W_{C,j}, \quad W_O = \sum_{k=1}^{K} W_{O,k}, \quad W_X = \sum_{j=1}^{J=K} W_{X,jj}$$

Non-Symmetric Cross Gramian [H. & Ohlberger'15 (Submitted)]:

$$\widetilde{W}_X := \sum_{j=1}^{J} \sum_{k=1}^{K} W_{X,jk}$$

- $\widetilde{W}_X \neq W_X \Rightarrow \widetilde{W}_X \neq W_C W_O$
- System is not required to be square, symmetric or gradient.
- Very efficient to compute as empirical non-symmetric cross gramian!