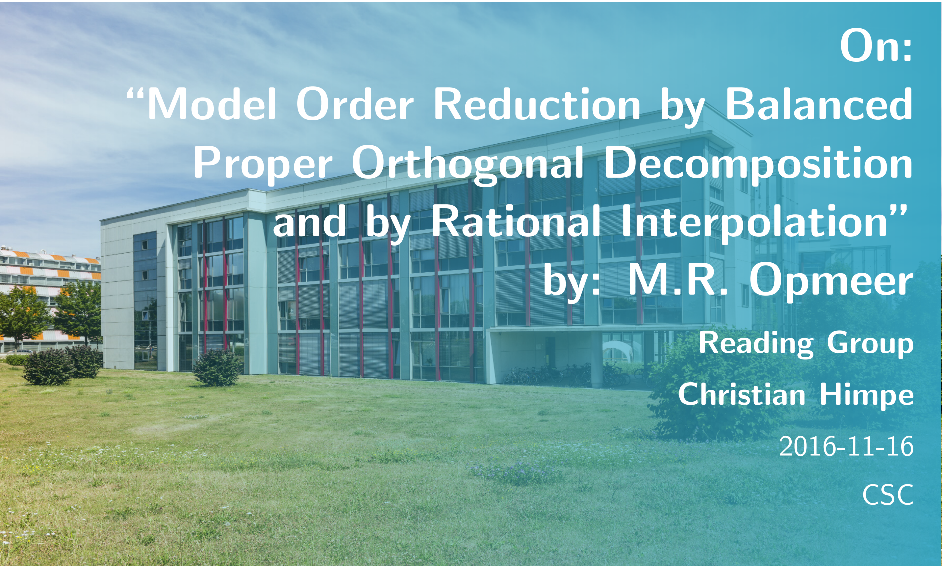
MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY



On: “Model Order Reduction by Balanced Proper Orthogonal Decomposition and by Rational Interpolation” by: M.R. Opmeer

Reading Group

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CSC

About

M.R. Opmeer. **Model Order Reduction by Balanced Proper Orthogonal Decomposition and by Rational Interpolation**. IEEE Transactions on Automatic Control 57(2): 472–477, 2012.

Main result:

- Rational inteprolation “is” balanced proper orthogonal decomposition.
- Number and step size of snapshots is related to interpolation points.

Outline

1. bPOD and RatInt
2. RatInt as bPOD
3. Two Examples

Model Order Reduction

Full Order Model:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t), \\
x(0) &= x_0.
\end{aligned}
$$

Reduced Order Model:

$$
\begin{aligned}
\dot{x}_r(t) &= A_r x_r(t) + B_r u(t), \\
y_r(t) &= C_r x_r(t) + Du(t), \\
x_r(0) &= x_{r,0}.
\end{aligned}
$$

Petrov-Galerkin Projection Operators:

$$
\begin{aligned}
S &: \mathbb{R}^N \to \mathbb{R}^n, \\
T &: \mathbb{R}^n \to \mathbb{R}^N, \\
T \circ S &= \mathbb{1}_n, \\
A_r &:= S \circ A \circ T, \\
B_r &:= S \circ B, \\
C_r &:= C \circ T, \\
x_{r,0} &:= S(x_0).
\end{aligned}
$$

For Now: SISO Systems

We assume a real, asymptotically stable SISO LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

with:

- $A \in \mathbb{R}^{N \times N}$,
- $B \in \mathbb{R}^{N \times 1}$,
- $C \in \mathbb{R}^{1 \times N}$,
- $D \in \mathbb{R}$.

Balanced Proper Orthogonal Decomposition

1. Numerically (i.e. by General Linear Methods) Compute:

$$\dot{x}(t) = Ax(t), \quad x(0) = B \quad \to \quad \mathcal{B} = [\tilde{x}(t_k)]_{k=1\dots K},$$
$$\dot{z}(t) = A^\intercal z(t), \quad z(0) = C^\intercal \quad \to \quad \mathcal{C} = [\tilde{z}(t_k)]_{k=1\dots K}.$$

2. Singular Value Decomposition of Empirical Hankel Operator:

$$\mathcal{H} := \mathcal{C}^\intercal \mathcal{B} \overset{\text{SVD}}{=} U \Sigma V.$$

3. Projection Operators by Method of Snapshots:

$$S := \Sigma^{-\frac{1}{2}} U^\intercal \mathcal{C}^\intercal,$$
$$T := \mathcal{B} V \Sigma^{-\frac{1}{2}}.$$

A sidenote concerning the empirical cross Gramian:

$$W_X := \mathcal{B}\mathcal{C}^\intercal \overset{\text{SVD}}{=} (u\Sigma^{\frac{1}{2}})(\Sigma^{\frac{1}{2}} v) = ST.$$

Rational Interpolation

1. Generalized controllability and observability operators:

$$\mathcal{R}(s) := [(s\mathbb{1} - A)^{-1}B, \dots, (s\mathbb{1} - A)^{-K}B], \quad \mathcal{R}(\infty) := [B, AB, \dots, A^{K-1}B],$$
$$\mathcal{O}(s) := [C(s\mathbb{1} - A)^{-1}, \dots, C(s\mathbb{1} - A)^{-K}]^\intercal, \quad \mathcal{O}(\infty) := [C, CA, \dots, CA^{K-1}]^\intercal.$$

2. Form operators V, W for $s_i \in \mathbb{C} \cup \infty$, $i = 1 \dots 2m$:

$$V := [\mathcal{R}(s_1), \dots, \mathcal{R}(s_m)],$$
$$W := [\mathcal{O}(s_{m+1}), \dots, \mathcal{O}(s_{2m})]^\intercal.$$

3. Projection operators are then given by:

$$S := (WV)^{-1}W,$$
$$T := V,$$

with $G_r(s_i) = G(s_i)$.

Interpolation at Infinity

Proposition:
Rational interpolation at $s = \infty$ yields the same reduced order models as balanced POD with samples obtained by the forward Euler (explicit) method.

Forward Euler Reminder ($k = 1 \dots K$):

$$
\begin{aligned}
&\tilde{x}(hk) = (\mathbb{1} + hA)^k B \\
&\rightarrow \mathcal{B} = [B, (\mathbb{1} + hA)B, \dots, (\mathbb{1} + hA)^{K-1} B], \\
&\rightarrow \mathcal{C} = [C^\intercal, (\mathbb{1} + hA^\intercal)C^\intercal, \dots, (\mathbb{1} + hA^\intercal)^{K-1} C^\intercal].
\end{aligned}
$$

A Short Justification

Proof:
For K samples, consider the upper triangular matrix $M \in \mathbb{R}^{K \times K}$:

$$M_{ij} := \binom{j-1}{i-1} h^{i-1} \Rightarrow \begin{cases} \mathcal{B} = \mathcal{R}(\infty) M \\ \mathcal{C} = \mathcal{O}(\infty)^* M \end{cases} \Rightarrow \mathcal{H} = M^\intercal \mathcal{O}(\infty) \mathcal{R}(\infty) M.$$

Projection operators:

$$S := \Sigma^{-\frac{1}{2}} U^\intercal M^\intercal \mathcal{O}(\infty),$$
$$T := \mathcal{R}(\infty) M V \Sigma^{-\frac{1}{2}}.$$

Let $Q := MV\Sigma^{-\frac{1}{2}}$ (being a similarity transformation), then:

$$\hat{S} := Q\Sigma^{-\frac{1}{2}} U^\intercal M^\intercal \mathcal{O}(\infty) = M\mathcal{H}^{-1} M^\intercal \mathcal{O}(\infty) = (\mathcal{O}(\infty)\mathcal{R}(\infty))^{-1} \mathcal{O}(\infty),$$
$$\hat{T} := \mathcal{R}(\infty) M V \Sigma^{-\frac{1}{2}} Q^{-1} = \mathcal{R}(\infty).$$

A 3×3 Example

Let's look at an example with 3 samples (remember the Pascal triangle?):

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & h & 2h \\ 0 & 0 & h^2 \end{pmatrix},$$

then the reachability factor of the empirical Hankel operator is:

$$\begin{aligned} \mathcal{R}(\infty)M &= \begin{pmatrix} B & AB & A^2B \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & h & 2h \\ 0 & 0 & h^2 \end{pmatrix} \\ &= \begin{pmatrix} B & (\mathbb{1} + hA)B & (\mathbb{1} + hA)^2 B \end{pmatrix} = \mathcal{B}, \end{aligned}$$

due to the binomial coefficent properties.

The same argument holds for the observability factor ...

Interpolation at a Finite Point

Proposition:
Rational interpolation at $s < \infty$ yields the same reduced order models as balanced POD with samples obtained by the backward Euler (implicit) method.

Backward Euler Reminder ($k = 1 \dots K$):

$$\begin{aligned} \tilde{x}(hk) &= (\mathbb{1} - hA)^{-k} B \\ \to \mathcal{B} &= [(\mathbb{1} - hA)^{-1} B, \dots, (\mathbb{1} - hA)^{-K} B], \\ \to \mathcal{C} &= [(\mathbb{1} - hA^\intercal)^{-1} C^\intercal, \dots, (\mathbb{1} - hA^\intercal)^{-K} C^\intercal]. \end{aligned}$$

Frequency and Stepwidth

Proof:
For K samples, consider the diagonal matrix $M \in \mathbb{R}^{K \times K}$:

$$M_{ii} := h^{-i} \Rightarrow \begin{cases} \mathcal{B} = \mathcal{R}(h^{-1})M \\ \mathcal{C} = \mathcal{O}(h^{-1})^* M \end{cases} \Rightarrow \mathcal{H} = M^\intercal \mathcal{O}(h^{-1})\mathcal{R}(h^{-1})M.$$

Similarly, we obtain:

$$\hat{S} = (\mathcal{O}(h^{-1})\mathcal{R}(h^{-1}))^{-1}\mathcal{O}(h^{-1}),$$
$$\hat{T} = \mathcal{R}(h^{-1}).$$

corresponding to rational interpolation at $s = h^{-1}$.

Interpolation at Multiple Points

Multiple points by joining snapshot sets:

- Forward Euler snapshots for $s_\infty = \infty$: $\mathcal{B}_\infty$
- Backward Euler snapshots for $s_i < \infty$: $\mathcal{B}_i$

$$\mathcal{B} := [\mathcal{B}_1, \dots, \mathcal{B}_K, \mathcal{B}_\infty]$$

Adapt M accordingly.

Proceed similarly for the adjoint system.

Interpolation at Complex Points

Look at the stability functions of the Euler methods:

- Forward Euler: $\Phi(z) = 1 + z$,
- Backward Euler: $\Phi(z) = \frac{1}{1-z}$.

The associated pole location relates to the interpolation points.

Higher order methods can produce the same results for larger time steps:

- Crank-Nicholson: $\Phi(z) = \frac{1+0.5z}{1-0.5z}$.

In this case twice the step size.

For complex interpolation points one could use for example:

- Hammer-Hollingsworth: $\Phi(z) = \frac{-12+6z+z^2}{12-6z+z^2}$.

The reciprocal of the poles determine the interpolation points.

For the Runge-Kutta SSPx2 method the stability function is: $\Phi(z) = \sum_{s=1}^{x} s^{-1} z^s$

Extension to MIMO Systems

For MIMO systems, bPOD is similar to tangential interpolation (TanInt):

- **Balanced POD for MIMO**:
  Sample for each column of B and row of C
  enlarging $\mathcal{B}$ and $\mathcal{C}$.
- **Tangential interpolation**:
  Rational interpolation using linear combinations of B and C
  yielding SISO systems.
- **Matrix interpolation**:
  Rational interpolation using all columns and rows of B and C
  (square systems only).

Examples

SISO Example:

- FEM for 1D Heat equation $\frac{\partial w}{\partial t} = \frac{\partial w^2}{\partial x^2}$, $(x,t) \in (0,1) \times \mathbb{R}_{>0}$,
  $w(0,x) = 0, \quad \frac{\partial w}{\partial x}(t,0) = u(t), \quad w(t,1) = 0, \quad y(t) = -w(t,0)$
- bPOD vs RatInt
- Tested: FE, BE, HH, CN

MISO Example:

- FEM for 1D Heat equation $\frac{\partial w}{\partial t} = \frac{\partial w^2}{\partial x^2} + w_2(t)\delta_{2/3}(x), \dots$
- bPOD vs TanInt
- Tested: BE

tl;dl

Why do I like this article?

- Connection between time- and frequency-domain and
- empirical Hankel operator with generalized operators.
- This somewhat extends to empirical Gramians.

Read: 10.1109/TAC.2011.2164018