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FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
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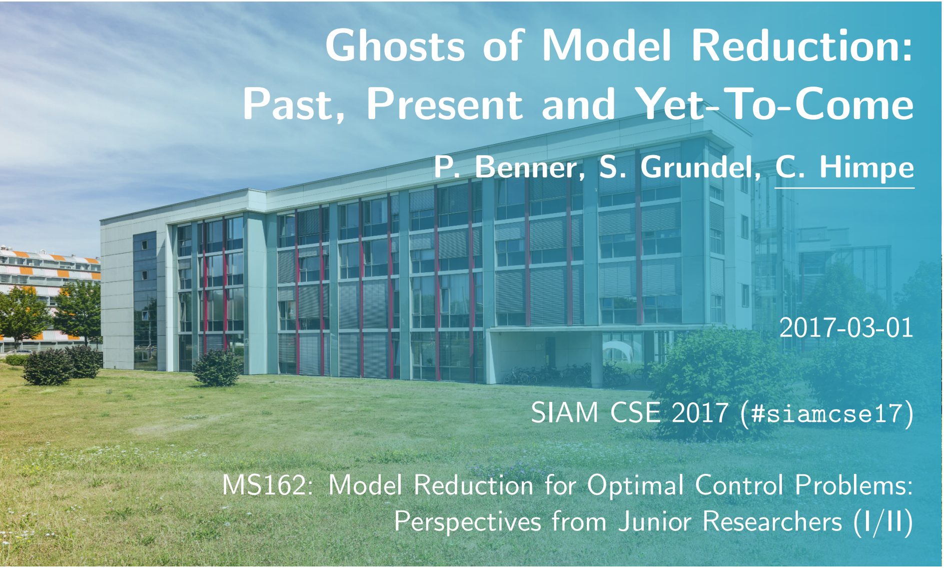
CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY



Ghosts of Model Reduction: Past, Present and Yet-To-Come

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MS162: Model Reduction for Optimal Control Problems:
Perspectives from Junior Researchers (I/II)

Outline

1. Past Ghosts
2. Present Ghosts
3. Future Ghosts

Disclaimer:

- Gramian-Based only!
- Only a subset!
- Purely subjective!

The Model

Input-Output Model:

$$
\begin{aligned}
E\dot{x}(t) &= f(t, x(t), u(t), \theta) \\
y(t) &= g(t, x(t), u(t), \theta) \\
x(0) &= x_0
\end{aligned}
$$

- Input: $u : \mathbb{R} \to \mathbb{R}^M$
- State: $x : \mathbb{R} \to \mathbb{R}^N$
- Output: $y : \mathbb{R} \to \mathbb{R}^Q$
- Parameter: $\theta \in \mathbb{R}^P$
- Mass Matrix: $E \in \mathbb{R}^{N \times N}$
- Vector Field: $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^N$
- Output Functional: $g : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$

From Inputs to Outputs

Input-Output-Mapping:

$$u \mapsto x(\theta) \mapsto y(\theta)$$

Model Reduction Idea:

$$u \mapsto x_r(\theta_r) \mapsto y_r(\theta_r)$$

Properties:

- $\dim(x_r(t)) \ll \dim(x(t))$
- $\dim(\theta_r) \ll \dim(\theta)$
- $\|y(\theta) - y_r(\theta_r)\| \ll 1$

Reduced Order Model:

$$\begin{aligned} E_r \dot{x}_r(t) &= f_r(t, x_r(t), u(t), \theta_r) \\ y_r(t) &= g_r(t, x_r(t), u(t), \theta_r) \\ x_r(0) &= x_{r,0} \end{aligned}$$

- Reduced State: $x_r : \mathbb{R} \to \mathbb{R}^n$
- Approximate Output: $y_r : \mathbb{R} \to \mathbb{R}^Q$
- Reduced Parameter: $\theta \in \mathbb{R}^p$
- Reduced Mass Matrix: $E_r \in \mathbb{R}^{n \times n}$
- Reduced Vector Field: $f_r : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^M \times \mathbb{R}^p \to \mathbb{R}^n$
- Reduced Output Functional: $g_r : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^M \times \mathbb{R}^p \to \mathbb{R}^Q$

Projection-Based Model Reduction

Reduced Order Model:

$$
\begin{aligned}
(V_1 E U_1)\dot{x}_r(t) &= V_1 f(t, U_1 x_r(t), u(t), \Pi_1 \theta_r) \\
y_r(t) &= \quad g(t, U_1 x_r(t), u(t), \Pi_1 \theta_r) \\
x_r(0) &= V_1 x_{r,0} \\
\theta_r &= \Lambda_1 \theta
\end{aligned}
$$

Task:

- Find truncated state projections: $\{U_1, V_1\}$
- Find truncated parameter projections: $\{\Pi_1, \Lambda_1\}$
- Lifting Bottleneck & Hyper-Reduction ?

Gramian-Based Model Reduction

- Based on System Theory[1] [2] [3]
- Input-Output systems properties:
  - **Controllability**
  - **Observability**
- Properties have associated operators
- Model reduction idea:
  - neither controllable
  - nor observable

  are discardable.
- Properties encoded in system Gramians

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[1] R.E. Kalman. **Mathematical Description of Linear Dynamical Systems**. Journal of the Society for Industrial and Applied Mathematics Series A Control, 1(2), 152–192. DOI 10.1137/0301010.

[2] T. Kailath. **Linear systems**. Information and System Sciences Series, Prentice-Hall, 1980.

[3] B.A. Francis. **A Course in H_∞ Control Theory**. Lecture Notes in Control and Information Sciences, vol 88, Springer, 1987. DOI 10.1007/BFb0007371.

I. Past Ghosts

1. Linear Systems
2. Reduced Order Model Quality
3. Single Operator Encoding

Linear Systems

Linear Time-Invariant System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

- Balanced Truncation[4]
- Square Root Algorithm[5]
- SVD-Based Balancing[6]

[4] B. Moore. **Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction**. IEEE Transactions on Automatic Control, 26(1): 17–32, 1981. DOI 10.1109/TAC.1981.1102568.

[5] M.S. Tombs and I. Postlethwaite. **Truncated balanced realization of stable non-minimal state-space system**. International Journal of Control, 46(4): 1319–1330, 1987. DOI 10.1080/00207178708933971.

[6] P. Benner, E.S. Quintana-Orti and G. Quintana-Orti. **Balanced Truncation Model Reduction of Large-Scale Dense Systems on Parallel Computers**. Mathematical and Computer Modelling of Dynamical Systems, 6(4): 383–405, 2000. DOI 10.1076/mcmd.6.4.383.3658.

Balanced Truncation

Input-Output Coherence via Hankel Singular Values:

$$H := \mathcal{O}\mathcal{C}$$

Controllability Gramian and Observability Gramian:

$$W_C := \mathcal{C}\mathcal{C}^* = \int_0^\infty \mathrm{e}^{At}\, BB^\intercal\, \mathrm{e}^{A^\intercal t}\,\mathrm{d}t, \quad W_O := \mathcal{O}^*\mathcal{O} = \int_0^\infty \mathrm{e}^{A^\intercal t}\, C^\intercal C\, \mathrm{e}^{At}\,\mathrm{d}t$$

Balancing Variant:

$$W_C \overset{\text{SVD}}{=} U_C D_C V_C, \quad W_O \overset{\text{SVD}}{=} U_O D_O V_O$$
$$UDV \overset{\text{SVD}}{=} U_C D_C^{\frac{1}{2}} V_C U_O D_O^{\frac{1}{2}} V_O = W_C^{\frac{1}{2}} W_O^{\frac{1}{2}}$$

- H_∞-Norm Error Bound[7] [8]
- Balanced Gains[9]
- Hilbert-Schmidt-Norm Optimality[10]

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[7] K. Glover. **All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds**. International Journal of Control, 39(6):1115–1193, 1984. DOI 10.1080/00207178408933239.

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H_∞- / L_2-Norm Error Bound:

$$\|G - G_r\|_{H_\infty} \leq 2 \sum_{i=n+1}^{N} \sigma_i(H) \rightarrow \|y - y_r\|_{L_2} \leq 2\|u\|_{L_2} \sum_{i=n+1}^{N} \sigma_i(H)$$

Balanced Gains:

instead of: $\sigma_i(H)$,

use: $d_i = \tilde{b}_i \tilde{b}_i^\intercal \sigma_i(H) = \tilde{c}_i^\intercal \tilde{c}_i \sigma_i(H) = |\tilde{b}_i \tilde{c}_i| \sigma_i(H)$

Single Operator

Cross Gramian:

$$W_X := \mathcal{C}\mathcal{O} = \int_0^\infty \mathrm{e}^{At} BC \, \mathrm{e}^{At} \, \mathrm{d}t$$

- SISO Systems[11]
- Symmetric MIMO Systems[12]
- MIMO Systems[13] [14] [15]

[11] K.V. Fernando and H. Nicholson. **On the Structure of Balanced and Other Principal Representations of SISO Systems**. IEEE Transactions on Automatic Control, 28(2): 228–231, 1983. DOI 10.1109/TAC.1983.1103195.

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Single Operator Encoding

Symmetric System:

$$H = H^* \Leftrightarrow C\, \mathrm{e}^A\, B = (C\, \mathrm{e}^A\, B)^\intercal$$

Orthogonally-Symmetric System:

$$P = P^\intercal : AP = PA^\intercal \wedge \exists U : UU^\intercal = \mathbb{1}, \begin{cases} B = PCU^\intercal & Q \leq M \\ C = PBU^\intercal & M \leq Q \end{cases}$$

$$\Rightarrow \widetilde{W}_X := \int_0^\infty \mathrm{e}^{At}\, BUC\, \mathrm{e}^{At}\, \mathrm{d}t$$

Approximate Cross Gramian by Embedding:

$$P = P^\intercal : AP = PA^\intercal \to \widetilde{B} := \begin{pmatrix} PC^\intercal & B \end{pmatrix}, \quad \widetilde{C} := \begin{pmatrix} C \\ B^\intercal P^{-1} \end{pmatrix}$$

Non-Symmetric Cross Gramian:

$$W_Z := \sum_{i=1}^{M} \sum_{j=1}^{Q} \int_0^\infty \mathrm{e}^{At}\, b_i c_j\, \mathrm{e}^{At}\, \mathrm{d}t$$

Other Past Ghosts

- Stability Preservation
- Structure Preservation
- Unstable Systems
- ...

II. Present Ghosts

1. Nonlinear Systems
2. Combined State and Parameter Reduction
3. Hyperbolic Systems

Nonlinear Systems

Control-Affine Nonlinear System:

$$\dot{x}(t) = f(x(t)) + h(x)u(t)$$
$$y(t) = g(x(t))$$

- Linearization[16]
- Nonlinear Balancing[17]
- Empirical Balanced Truncation[18] & Empirical Cross Gramian[19]

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[16] X. Ma and J.A. De Abreu-Garcia. **On the Computation of Reduced Order Models of Nonlinear Systems using Balancing Technique**. In Proceedings of the 27th IEEE Conference on Decision and Control, vol 2: 1165–1166, 1988. DOI 10.1109/CDC.1988.194502.

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Empirical Cross Gramians

Linear Empirical Cross Gramian:

$$W_X = \int_0^\infty (e^{At} B)(C e^{At}) \, dt = \int_0^\infty (e^{At} B)(e^{A^\intercal t} C^\intercal)^\intercal \, dt = \int_0^\infty x(t)(z(t))^\intercal \, dt$$

Nonlinear Empirical Cross Gramian:

$$\widehat{W}_X := \frac{1}{M} \sum_{m=1}^{M} \int_0^\infty \Psi^m(t) \, dt$$

$$\Psi_{ij}^m(t) = (x_i^m(t) - \bar{x}_i^m)(y_m^j(t) - \bar{y}_m^j)$$

Nonlinear Empirical Non-Symmetric Cross Gramian:

$$\widehat{W}_X := \frac{1}{M} \sum_{q=1}^{Q} \sum_{m=1}^{M} \int_0^\infty \Psi^{mq}(t) \, dt$$

$$\Psi_{ij}^{mq}(t) = (x_i^m(t) - \bar{x}_i^m)(y_q^j(t) - \bar{y}_q^j)$$

Combined Reduction

Parametric System:

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), \theta) \\
y(t) &= g(x(t), u(t), \theta) \\
\dim(x(t)) &\gg 1 \\
\dim(\theta) &\gg 1
\end{aligned}
$$

- Active Subspaces[20]
- Greedy Combined Reduction[21]
- Gramian-Based Combined Reduction[22]

[20] P.G. Constantine. **Active Subspaces: Emerging Ideas for Dimension Reduction in Parameter Studies**. SIAM Spotlights, SIAM, 2015. DOI 10.1137/1.9781611973860.

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Augmented system:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} f(x(t), u(t), \theta(t)) \\ 0 \end{pmatrix}$$

$$y(t) = g(x(t), u(t), \theta(t))$$

$$\begin{pmatrix} x(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix}$$

Empirical cross Gramian of augmented system (joint Gramian):

$$W_J = \begin{pmatrix} W_X & W_M \\ W_m & W_P \end{pmatrix} = \begin{pmatrix} W_X & W_M \\ 0 & 0 \end{pmatrix}$$

Cross-Identifiability Gramian:

$$W_{\ddot{I}} := -\frac{1}{2} W_M^\intercal (W_X + W_X)^{-1} W_M$$

- Wave Centering[23]
- Energy-Stable Inner Products[24] [25]
- Method of Freezing[26]

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[23] S. Glavaski, J.E. Marsden and R.M. Murray. **Model Reduction, Centering, and the Karhunen-Loeve Expansion**. Proceedings of the 37th IEEE Conference on Decision and Control: 2071–2076, 1998. DOI 10.1109/CDC.1998.758639.

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Treating Hyperbolicity

Wave Centering:

$$\int_0^{\frac{1}{2}} [f(x-c)]^2 \, \mathrm{d}x = \int_{\frac{1}{2}}^1 [f(x-c)]^2 \, \mathrm{d}x$$

$$\rightarrow x(t) = \sum_{i=1}^{N} a_i(t) \varphi_i(x + c(t)) + x^c(x + c(t))$$

Stability-Preserving Lyapunov Inner Product:

$$\dot{x}(t) = Ax(t)$$

$$\lambda(A) < 0 \Rightarrow \exists V(x) = x^\intercal P x, \quad AP + PA = Q, Q = Q^\intercal > 0$$

Other Present Ghosts

- Parametric
- Switched Systems
- Delay Systems
- Descriptor Systems
- Bilinear / Quadratic
- Time-Varying (Parameter)
- ...

1. Holistic Reduction[27]
2. General Nonlinear Systems[28]
3. Kernel Methods[29]

[27] E.B. Le, A. Myers and T. Bui-Thanh. **A Triple Model Reduction for Data-Driven Large-Scale Inverse Problems in High Dimensional Parameter Spaces**. SIAM Conference on Uncertainty Quantification, MS119 Reduced-order Modeling in Uncertainty Quantification - Part III, 2016. URL http://meetings.siam.org/sess/dsp_talk.cfm?p=74883.

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Bringing it All Together

Cross-Operator-Based Reduction:

- Empirical Cross Gramian
- Non-Symmetric Variant (Free Stability)
- Joint Gramian (Cross-Identifiability Gramian)
- Partitioned Computation[30] (Distributed Empirical Cross Gramian)
- SVD-Based Balancing (Approximate Balancing)
- Partitioned Decomposition via HAPOD[30]

[30] C. Himpe, T. Leibner and S. Rave. **Hierarchical Approximate Proper Orthogonal Decomposition**. Preprint, arXiv math.NA: 1607.05210, 2016. URL http://arxiv.org/abs/1607.05210.

My Previous Ghost: Brain Connectivity

A Tough Conglomeration:

- Nonlinear
- Parametric
- Stability-Preserving
- Combined State and Parameter Reduction

The empirical cross Gramian (empirical joint Gramian) works here[31]!

[31] C. Himpe. **Combined State and Parameter Reduction with an Application in Neuroscience**. Westfälische Wilhelms-Universität Münster, 2016.

My Current Ghost: Gas Networks

A Diabolical Concoction:

- Nonlinear
- Parametric
- Hyperbolic
- Descriptor System
- Structure-Preserving
- Stability-Preserving
- Combined State and Parameter Reduction
- (possibly Switched and Non-Smooth)

A Gas Transportation Network Model

Implicit ODE Form[32]:

$$E\dot{x}(t) = Ax(t) + f(x(t), u(t), \theta) + Bu(t)$$
$$y(t) = Cx(t)$$

Zooming In:

$$\begin{pmatrix} E_p & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} 0 & A_q \\ A_p & 0 \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} + \begin{pmatrix} \dot{u}_s(t) \\ f_q(p(t), q(t), u_s(t), \theta) \end{pmatrix} + \begin{pmatrix} B_p \\ B_q \end{pmatrix} \begin{pmatrix} u_d(t) \\ u_s(t) \end{pmatrix}$$
$$\begin{pmatrix} y_d(t) \\ y_s(t) \end{pmatrix} = \begin{pmatrix} C_p \\ C_q \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

[32] S. Grundel, L. Jansen, N. Hornung, T. Clees, C. Tischendorf and P. Benner. **Model Order Reduction of Differential Algebraic Equations Arising from the Simulation of Gas Transport Networks**. In Progress in Differential-Algebraic Equations, Springer: 183–205, 2014. DOI 10.1007/978-3-662-44926-4_9.

Model Reduction for Gas Networks

Part of the MathEnergy[33] Project.

General Approach:

- Data-Driven
- Structure-Preserving
- Index-Reduction

Considered Methods:

- Proper Orthogonal Decomposition / balanced POD
- Dynamic Mode Decomposition / ioDMD
- **Empirical Balanced Truncation / Empirical Cross Gramian**

[33] MathEnergy is funded by the BMWi.

Empirical Gramians:

- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Linear Cross Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

Features:

- Interfaces for: Solver, inner product kernels & distributed memory
- Non-Symmetric option for all cross Gramians
- Compatible with OCTAVE and MATLAB
- Vectorized and parallelizable
- Open-source licensed

More info: `http://gramian.de`

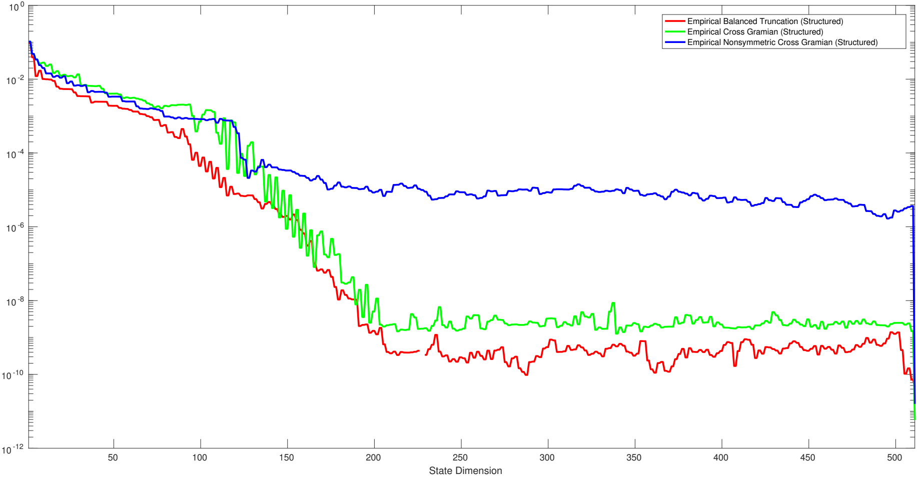
C. Himpe. **emgr - The empirical Gramian Framework**. Submitted, Preprint: arXiv (cs.MS): 1611.00675, 2016.

Test System

Gas Network Model w. Simple Topology

- 1 Supply node
- 1 Demand node
- 256 Network nodes
- Input: supply pressure and demand flow
- State: $2 \times 256 = 512$ states
- Output: supply flow and demand pressure
- Time Horizon: $24 \times 60 = 1440$ timesteps

Preliminary Numerical Experiments



So why the cross Gramian?

- Always a square system
- Low-communication parallelization
- Less offline time for larger models
- Robust parameter identification / reduction

Summary

- Remixing balancing technologies
- Data-driven & averaging for nonlinear systems
- Empirical cross Gramian for gas networks

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