MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

The Cross Gramian

An Overview and Open Problems

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Outline

1. Obligatory Notation
2. Cross Gramian Flavors
3. Cross Gramian Related Open Problems
    I. Galerkin projection error bound
    II. $\mathcal{H}_2$ optimized cross Gramian
    III. Nonlinear cross Gramians
4. Cross Gramians for Gas Transport

Input-Output Systems

Nonlinear Parametric Input-Output Systems:

$$\dot{x}(t) = f(x(t), u(t), \theta)$$
$$y(t) = g(x(t), u(t), \theta)$$

Linear Input-Output System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

- $M := \dim(u(t))$
- $N := \dim(x(t))$
- $Q := \dim(y(t))$
- $P := \dim(\theta)$

Reduced Order Model

Reduced Nonlinear Input-Output Systems:

$$\begin{aligned} \dot{x}_r(t) &= f_r(x_r(t), u(t), \theta_r) \\ y_r(t) &= g_r(x_r(t), u(t), \theta_r) \end{aligned}$$

Reduced Linear Input-Output System:

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u(t) \\ y_r(t) &= C_r x_r(t) \end{aligned}$$

- $n := \dim(x_r(t)) \ll \dim(x(t))$
- $p := \dim(\theta_r) \ll \dim(\theta)$
- $\|y(\theta) - y_r(\theta_r)\| \ll 1$

Projection-Based Model Reduction

Reduced Nonlinear Input-Output Systems:

$$\begin{aligned} \dot{x}_r(t) &= V_1 f(U_1 x_r(t), u(t), \Pi_1 \theta_r) \\ y_r(t) &= \quad g(U_1 x_r(t), u(t), \Pi_1 \theta_r) \end{aligned}$$

Reduced Linear Input-Output System:

$$\begin{aligned} \dot{x}_r(t) &= (V_1 A U_1) x_r(t) + (V_1 B) u(t) \\ y_r(t) &= (C U_1) x_r(t) \end{aligned}$$

- $U_1 \in \mathbb{R}^{N \times n}$, $V_1 \in \mathbb{R}^{n \times N}$, $V_1 U_1 = \mathbb{1}$, $x_r(t) = V_1 x(t)$
- $\Pi_1 \in \mathbb{R}^{P \times p}$, $\Lambda_1 \in \mathbb{R}^{p \times P}$, $\Lambda_1 \Pi_1 = \mathbb{1}$, $\theta_r = \Lambda_1 \theta$
- Hyperreduction is a different story.

Hankel Operator

Evolution Operator (infinite rank!)[1]

$$S(u) := C \int_0^\infty \mathrm{e}^{At} Bu(t)\mathrm{d}t$$

Controllability Operator:

$$\mathcal{C}(u) := \int_0^\infty \mathrm{e}^{At} Bu(-t)\mathrm{d}t$$

Observability Operator:

$$\mathcal{O}(x_0) := C \,\mathrm{e}^{At} x_0$$

Hankel Operator (finite rank!)[2]:

$$H := \mathcal{O} \circ \mathcal{C}$$

[1] A.C. Antoulas. **Approximation of Large-Scale Dynamical Systems**. Vol. 6 of Advances in Design and Control, SIAM, 2005.

[2] B.A. Francis. **A Course in H_∞ Control Theory**. Vol. 88 of Lecture Notes in Control and Information Sciences, Springer, 1987.

Cross Gramian Matrix

Hankel Operator (maps past inputs to future outputs):

$$H := \mathcal{O} \circ \mathcal{C}$$

Cross Gramian[3] (note it's generally not a Gramian matrix!):

$$W_X := \mathcal{C} \circ \mathcal{O} = \int_0^\infty \mathrm{e}^{At} \, BC \, \mathrm{e}^{At} \, \mathrm{d}t$$

$$\Leftrightarrow \; AW_X + W_X A = -BC$$

- $\lambda_i(A) < 0$
- $M \stackrel{!}{=} Q$
- $\mathrm{tr}(W_X) = \mathrm{tr}(H)$
- $W_X : \mathbb{R}^N \to \mathbb{R}^N$

[3] K.V. Fernando. **Covariance and Gramian matrices in control and systems theory**. University of Sheffield, 1983.

Relation to Balanced Truncation

Symmetric System:

$$\mathcal{O}\mathcal{C} = (\mathcal{O}\mathcal{C})^* \Rightarrow W_X^2 = \mathcal{C}\mathcal{O}\mathcal{C}\mathcal{O} = \mathcal{C}\mathcal{C}^*\mathcal{O}^*\mathcal{O} = W_C W_O$$

Cross Gramian is equivalent to balanced truncation.

State-Space Symmetric System:

$$A = A^\intercal, \quad C = B^\intercal \Rightarrow \mathcal{C}\mathcal{O} = \mathcal{C}\mathcal{C}^* = \mathcal{O}^*\mathcal{O}$$

All system Gramians are equal.

Approximate Balancing

Approximate balancing[4] via singular value decomposition:

$$W_X \overset{\text{SVD}}{=} UDV$$

Direct Truncation (Galerkin projection):

$$U_1 := U_{:,1:n}, \quad \sum_{i=1}^{n} D_{ii} < \varepsilon$$
$$V_1 := U_1^\intercal$$

[4] D.C. Sorensen and A.C. Antoulas. **The Sylvester equation and approximate balanced reduction**. Linear Algebra and its Applications, 351–352:671–700, 2002.

Non-Symmetric Cross Gramian

Cross Gramian of a square MIMO as sum of SISOs:

$$W_X = \sum_{i=1}^{M} \int_0^\infty \mathrm{e}^{At}\, B_{:,i} C_{i,:}\, \mathrm{e}^{At}\, \mathrm{d}t$$

Non-Symmetric Cross Gramian[5] (Cross Gramian of average system):

$$W_Z := \sum_{i=1}^{M} \sum_{j=1}^{Q} \int_0^\infty \mathrm{e}^{At}\, B_{:,i} C_{j,:}\, \mathrm{e}^{At}\, \mathrm{d}t$$

$$= \int_0^\infty \mathrm{e}^{At} \Big(\sum_{i=1}^{M} B_{:,i} \Big) \Big(\sum_{j=1}^{Q} C_{j,:} \Big) \mathrm{e}^{At}\, \mathrm{d}t$$

- Motivated by Decentralized Control
- Stability Preserving (since all SISO systems are symmetric)

[5] C. H. and M. Ohlberger; **A note on the cross gramian for non-symmetric systems**. System Science and Control Engineering 4(1): 199–208, 2016

Empirical Linear Cross Gramian

Primal-Dual System:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^\intercal \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} B \\ C^\intercal \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

$$\rightarrow \overline{W}_C = \begin{pmatrix} W_C & W_X \\ W_X^\intercal & W_O \end{pmatrix}$$

- Primal Impulse Response: $g_x(t) = \mathrm{e}^{At} B$
- Dual Impulse Response: $g_z(t) = \mathrm{e}^{A^\intercal t} C^\intercal$

Empirical Linear Cross Gramian[6]:

$$W_X = \int_0^\infty (\mathrm{e}^{At} B)(\mathrm{e}^{A^\intercal t} C^\intercal)^\intercal \mathrm{d}t \approx \int_0^\infty x(t) z(t)^\intercal \mathrm{d}t =: W_Y$$

[6] U. Baur, P. Benner, B. Haasdonk, C. H., I. Martini and M. Ohlberger. **Comparison of Methods for Parametric Model Order Reduction of Time-Dependent Problems**. In: Model Reduction and Approximation: Theory and Algorithms, Editors: P. Benner, A. Cohen, M. Ohlberger and K. Willcox, SIAM, 2017.

Empirical Cross Gramian

Empirical Cross Gramian[7]:

$$\widehat{W}_X := \frac{1}{M} \sum_{m=1}^{M} \int_0^\infty \Psi^m(t) \mathrm{d}t \in \mathbb{R}^{N \times N}$$

$$\Psi_{ij}^m(t) = (x_i^m(t) - \bar{x}_i^m)(y_m^j(t) - \bar{y}_m^j) \in \mathbb{R}$$

- $x^i(t)$ is a state trajectory with a perturbed i-th input.
- $y^m(t)$ is an output trajectory with a perturbed m-th initial state.
- Applicable to nonlinear systems: only $x^i(t)$ and $y^m(t)$ required.
- Equal to linear cross Gramian for linear systems.
- Efficient empirical non-symmetric cross Gramian.

[7] C. H. and M. Ohlberger. **Cross-Gramian Based Combined State and Parameter Reduction for Large-Scale Control Systems**. Mathematical Problems in Engineering 2014: 1–13, 2014.

Empirical Joint Gramian

Augmented System:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} f(x(t), u(t), \theta(t)) \\ 0 \end{pmatrix}$$
$$y(t) = g(x(t), u(t), \theta(t))$$

Joint Gramian[4] (Empirical Cross Gramian of the Augmented System):

$$W_J = \begin{pmatrix} W_X & W_M \\ 0 & 0 \end{pmatrix}$$

Cross-Identifiability Gramian (Schur Complement of Symmetric Part of W_J):

$$W_{\ddot{I}} := -W_M^{\intercal} W_X^{-1} W_M$$

Empirical Gramians

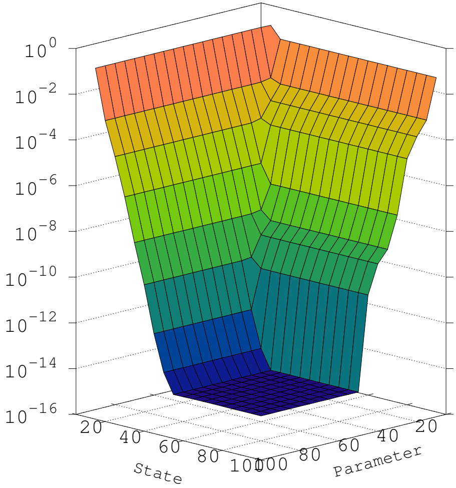
- Applicable to any system that can be simulated:
  - Nonlinear systems
  - Parametric systems
  - Time-varying systems
- Basic idea is averaging.
- Simple computation.
- Allows high-dimensional parameter spaces.
- Enables combined state and parameter reduction[8].

More info on empirical Gramians:

C. H. `emgr` - **The Empirical Gramian Framework**. arXiv cs.MS: 1611.00675, 2016.

[8] C. H. **Combined State and Parameter Reduction for Nonlinear Systems with an Application in Neuroscience**. Westfälische Wilhelms Universität, Sierke Verlag Göttingen, 2017.

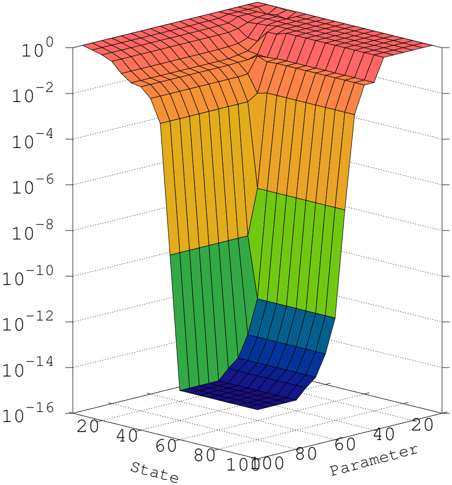
Combined Reduction I



Combined reducibility for the nonlinear RC cascade benchmark[9].

[9] MORwiki. **Nonlinear RC Ladder**. `http://modelreduction.org/index.php/Nonlinear_RC_Ladder`

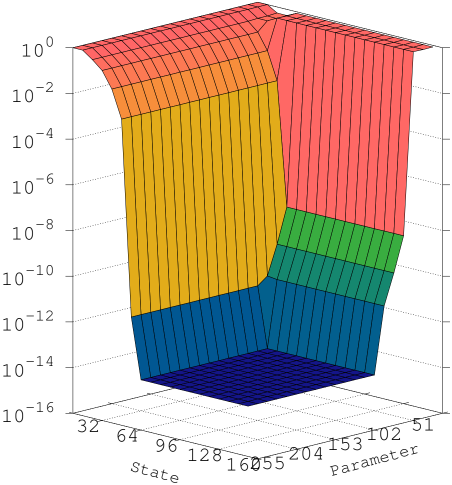
Combined Reduction II



Combined reducibility for the hyperbolic network model[10].

[10] Y. Quan, H. Zhang, and L. Cai. **Modeling and Control Based on a New Neural Network Model**. In Proceedings of the American Control Conference, volume 3, pages 1928–1929, 2001.

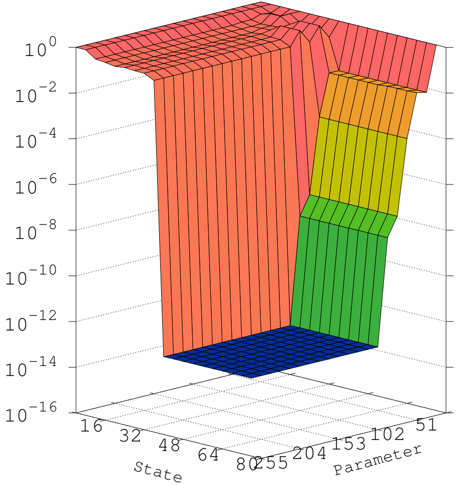
Combined Reduction III



Combined reducibility for the EEG dynamic causal model[11].

[11] O. David, S.J. Kiebel, L.M. Harrison, J. Mattout, J.M. Kilner, and K.J. Friston. **Dynamic causal modeling of evoked responses in EEG and MEG**. NeuroImage , 4: 1255–1272, 2006.

Combined Reduction IIII



Combined reducibility for the fMRI dynamic causal model[12]

[12] K.J. Friston, L.M. Harrison, and W. Penny. **Dynamic causal modelling**. NeuroImage 19(4):1273–1302, 2003.

(My) Open Problems

I. Direct Truncation Error Bound

II. $\mathcal{H}_2$ Optimized Cross Gramian

III. Empirical Cross Gramian vs Nonlinear Cross Gramian

(I.) Distributed Cross Gramian

Column-wise cross Gramian computation:

$$\widehat{W}_X = \begin{pmatrix} w_{X,1} & \dots & w_{X,N} \end{pmatrix}$$

$$w_{X,j} = \frac{1}{M} \sum_{m=1}^{M} \int_0^\infty \psi^{jm}(t) \mathrm{d}t \in \mathbb{R}^N$$

$$\psi_i^{jm}(t) = (x_i^m(t) - \bar{x}_i^m)(y_m^j(t) - \bar{y}_m^j) \in \mathbb{R}$$

- Only for empirical cross Gramians (W_X, W_Y, W_Z, W_J)!
- Overcome curse of dimensionality ($W_X \in \mathbb{R}^{N \times N}$).
- Hierarchical Approximate Proper Orthogonal Decomposition[13]
  - Direct distributed computation (of U_1)
  - Direct incremental computation (of U_1)
  - More on the HAPOD, (see S. Rave's talk on 2017-08-15, 12:00)

[13] C. H. and T. Leibner and S. Rave. **Hierarchical Approximate Proper Orthogonal Decomposition**. arXiv math.NA: 1607.05210, 2016.

Mean Projection Error Bound:

$$\|W_X - U_1 U_1^\intercal W_X\|_2 \leq \sqrt{\sum_{i=1}^{n} \sigma_i(W_X)^2}$$

State Error Bound[14]:

$$\|x(t) - x_r(t)\|_2 \leq c(\|x_0 - U_1 U_1^\intercal x_0\| + \int_0^\infty \|R(t)\|_2 \,\mathrm{d}t)$$

[14] B. Haasdonk and M. Ohlberger. **Efficient reduced models and a posteriori error estimation for parametrized dynamical systems by offline/online decomposition**. Mathematical and Computer Modelling of Dynamical Systems 17(2): 145–161, 2011.

(II.) Tangential Interpolation

Tangential Interpolation (using directions: r^i and l^j):

$$V_1 := \bigoplus_i \mathcal{C}(s_i) r^i, \quad U_1 := \bigoplus_j l^j \mathcal{O}(s_j).$$

Frequency Domain Cross Gramian:

$$W_X = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\imath\omega \mathbb{1} - A)^{-1} B C (\imath\omega \mathbb{1} - A)^{-1} \mathrm{d}\omega$$

Tangential Cross Gramian:

$$\begin{aligned} W_{X,rl} := (\mathcal{C}r)(l\mathcal{O}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\imath\omega \mathbb{1} - A)^{-1} B r l C (\imath\omega \mathbb{1} - A)^{-1} \mathrm{d}\omega \\ &= \int_0^{\infty} \mathrm{e}^{At} (Br)(lC)\, \mathrm{e}^{At}\, \mathrm{d}t \end{aligned}$$

$$\rightarrow r_i = l_j = 1 \; \forall\, i, j \Rightarrow W_{X,rl} = W_Z$$

Tangential Cross Gramian:

$$W_{X,rl} = \int_0^\infty \mathrm{e}^{At}\, BrlC\, \mathrm{e}^{At}\, \mathrm{d}t$$

- What are the "best" directions r and l?
- What are desirable properties of $BrlC$?
- Can (simplified) balanced gains[15] help:

$$d_i := |\tilde{b}_i \tilde{c}_i| \sigma_i(H)$$

[15] A.M. Davidson. **Balanced systems and model reduction**. Electronics Letters, 22(10): 531–532, 1986.

(III.) Nonlinear Cross Gramian

Control-Affine Nonlinear System:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y(t) &= h(x(t)) \end{aligned}$$

Nonlinear Cross Gramian[16] (Solution to a nonlinear Sylvester equation):

$$\frac{\partial \Phi}{\partial x} f(x) + f(\Phi(x)) = -g(\Phi(x))h(x)$$

[16] T.C. Ionescu, K. Fujimoto and J.M.A. Scherpen. **Singular value analysis of nonlinear symmetric systems**. IEEE Transactions on Automatic Control, 56(9): 2073–2086, 2011.

(III.) Empirical vs. Nonlinear

Explicit nonlinear cross Gramian definition:

$$\Phi(x_0) := \mathcal{C} \circ \mathcal{O}(x_0) = \ ?$$

$$\begin{aligned} \mathcal{C}(u) &= \chi(t), \ \ \dot{\chi}(t) = -f(\chi(t)) - g(\chi(t))u(t) \\ \mathcal{O}(t) &= h(x(t)), \ \ \dot{x}(t) = f(x(t)) \end{aligned}$$

- Is there an empirical formulation of the nonlinear cross Gramian?
- Is the empirical cross Gramian an approximation to the nonlinear?

Gas Transport in a Pipeline

1D (Simplified) Isothermal Euler Equations[17]:

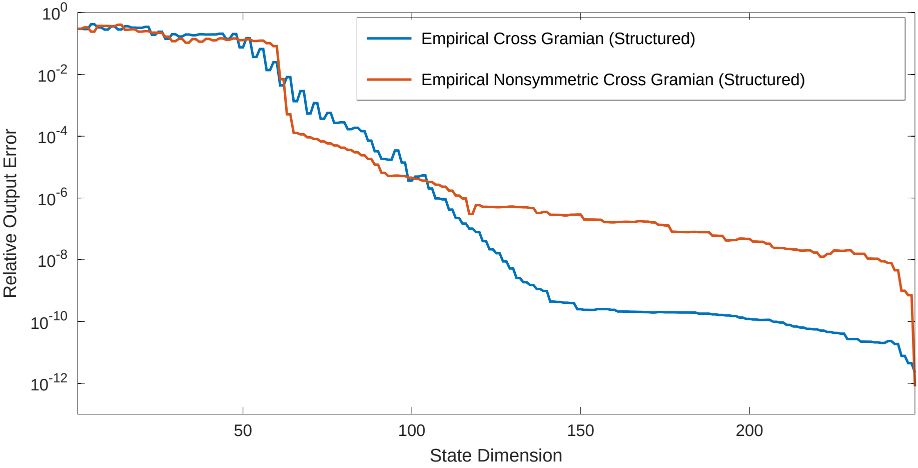
$$\frac{\partial p}{\partial t} = -\frac{\partial q}{\partial x}$$
$$\frac{\partial q}{\partial t} = -c^2 \frac{\partial p}{\partial x} - \frac{\lambda}{2D} \frac{q|q|}{p}$$

- System properties: hyperbolic, nonlinear, coupled.
- Finite difference spatial discretization: DAE.
- Analytic index reduction to implicit ODE.
- Structured projections[18]:
  - Pressure cross Gramian
  - Mass-flux cross Gramian

[17] S. Grundel, Jansen, N. Hornung, T. Clees, C. Tischendorf and P. Benner. **Model Order Reduction of Differential Algebraic Equations Arising from the Simulation of Gas Transport Networks**. In: Progress in Differential-Algebraic Equations: 183–205, 2014.

[18] T. Reis and T. Stykel. **Balanced truncation model reduction of second-order systems**. Mathematical and Computer Modelling of Dynamical Systems: Methods, Tools and Applications in Engineering and Related Sciences, 14(5): 391–406, 2008.

Preliminary Result



Summary

Cross-Gramian-Related Open Problems:

I. Direct Truncation Error Bound
II. $\mathcal{H}_2$ Optimized Cross Gramian
III. Empirical vs Nonlinear Cross Gramian

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