MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Fast Low-Rank Empirical Cross Gramians

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#gamm17

Outline

1. Model Reduction
2. Cross Gramian
3. Empirical Cross Gramian
4. Fast & Low Rank Empirical Cross Gramian
5. Numerical Example

Motivation

Mathematically:

- Fast simulation of large-scale, nonlinear models,
- based on established (dense) linear methods,
- exploiting distributed memory systems.

Practically (among many others):

- for mechanical systems,
- or dynamic network models,
- such as gas transportation networks.

Model

Input-Output System:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$
$$x(0) = x_0$$

Input:	$u : \mathbb{R} \to \mathbb{R}^M$
State:	$x : \mathbb{R} \to \mathbb{R}^N$
Output:	$y : \mathbb{R} \to \mathbb{R}^Q$
Vector field:	$f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$
Output functional:	$g : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^Q$

Model Reduction

Reduced Order Model (ROM):

$$\begin{aligned} \dot{x}_r(t) &= f_r(x_r(t), u(t)) \\ y_r(t) &= g_r(x_r(t), u(t)) \\ x_r(0) &= x_{r,0} \end{aligned}$$

Reduced state:	$x_r : \mathbb{R} \to \mathbb{R}^{n \ll N}$
Reduced output:	$y_r : \mathbb{R} \to \mathbb{R}^{Q}$
Model reduction error:	$\lVert y - y_r \rVert \ll 1$
Reduced vector field:	$f_r : \mathbb{R}^n \times \mathbb{R}^M \to \mathbb{R}^n$
Reduced output functional:	$g_r : \mathbb{R}^n \times \mathbb{R}^M \to \mathbb{R}^Q$

Projection-Based Model Reduction

Projection-Based Reduced Order Model:

$$\begin{aligned} \dot{x}_r(t) &= V_1 f(U_1 x_r(t), u(t)) \\ y_r(t) &= \quad g(U_1 x_r(t), u(t)) \\ x_r(0) &= V_1 x_0 \end{aligned}$$

Reconstructing truncated projection: $U_1 \in \mathbb{R}^{N \times n}$
Reducing truncated projection: $V_1 \in \mathbb{R}^{n \times N}$

Task: Find U_1 and V_1.

From Linear to General

Linear Time-Invariant System:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{aligned}$$

(Nonlinear) model reduction shopping list:

- Based on linear model reduction techniques.
- Capture features beyond linearization.
- Related to established balancing methods.
- Numerically simple.

Cross Gramian

Square Linear Time-Invariant System:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ \dim(\mathrm{vec}(B)) &= \dim(\mathrm{vec}(C)) \end{aligned}$$

Cross Gramian Matrix[1]:

$$W_X := \int_0^\infty \mathrm{e}^{At}\, BC\, \mathrm{e}^{At}\, \mathrm{d}t \in \mathbb{R}^{N \times N}$$

Computation via Sylvester Equation:

$$AW_X + W_X A = -BC$$

[1] K.V. Fernando and H. Nicholson. **On the Structure of Balanced and Other Principal Representations of SISO Systems**. IEEE Transactions on Automatic Control, 28(2): 228–231, 1983. DOI 10.1109/TAC.1983.1103195.

Balancing projection[2] (for symmetric systems only):

$$W_X \overset{\text{EVD}}{=} T \Lambda T^{-1} \to \begin{cases} T &= \begin{pmatrix} U_1 & U_2 \end{pmatrix} \\ T^{-1} &= \begin{pmatrix} V_1 & V_2 \end{pmatrix}^\intercal \end{cases}$$

Approximately balancing projection[3]:

$$W_X \overset{\text{SVD}}{=} UDV \to \begin{cases} U = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \\ V = \begin{pmatrix} V_1 & V_2 \end{pmatrix}^\intercal \end{cases}$$

[2] R.W. Aldhaheri. **Model order reduction via real Schur-form decomposition**. International Journal of Control, 53(3): 709–716, 1991. DOI 10.1080/00207179108953642.

[3] D.C. Sorensen and A.C. Antoulas. **The Sylvester equation and approximate balanced reduction**. Linear Algebra and its Applications, 351–352: 671–700, 2002. DOI 10.1016/S0024-3795(02)00283-5.

Empirical Gramians

Alternative System Gramians:

- Biliner-Quadratic Gramians[4]
- Nonlinear Gramians[5]
- **Empirical Gramians**[6]
- ...

Empirical Gramian Properties:

- Data-driven, based on (output) trajectory data
- Applicable to general nonlinear systems
- Reduce to linear Gramians for linear systems

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[4] P. Benner, P. Goyal and S. Gugercin. **$\mathcal{H}_2$-Quasi-Optimal Model Order Reduction for Quadratic-Bilinear Control Systems**. arXiv math.NA: 1610.03279, 2016. URL http://arxiv.org/pdf/1610.03279

[5] J.M.A. Scherpen. **Balancing for nonlinear systems**. Systems and Control Letters, 21(2): 143–153, 1993. DOI 0167-6911(93)90117-O.

[6] S. Lall, J.E. Marsden, and S. Glavaski. **Empirical Model Reduction of Controlled Nonlinear Systems**. In Proceedings of the 14th IFAC World Congress, vol F: 473–478, 1999. URL resolver.caltech.edu/CaltechAUTHORS:20101007-154754737.

Empirical Linear Cross Gramian

Empirical Linear Cross Gramian:

$$W_X = \int_0^\infty (\mathrm{e}^{At} B)(\mathrm{e}^{A^\intercal t} C^\intercal)^\intercal \, \mathrm{d}t$$

- Integrand is product of
  - impulse response and
  - adjoint impulse response.
- Cross Gramian only from trajectory data.
- **But**: No (easy) adjoint system for nonlinear systems.

Empirical Cross Gramian

Empirical Cross Gramian[7]:

$$\widehat{W}_X := \frac{1}{M} \sum_{m=1}^{M} \int_0^\infty \Psi^m(t)\,\mathrm{d}t \in \mathbb{R}^{N \times N}$$

$$\Psi_{ij}^m(t) = (x_i^m(t) - \bar{x}_i^m)(y_m^j(t) - \bar{y}_m^j) \in \mathbb{R}$$

- Applicable to nonlinear (square) systems.
- For asymptotically stable linear systems: $\widehat{W}_X = W_X$.
- Related to (empirical) balanced truncation,
- and balanced POD (bPOD).
- Non-symmetric variant[8] for $\dim(u(t)) \neq \dim(y(t))$.

[7] C. Himpe and M. Ohlberger. **Cross-Gramian Based Combined State and Parameter Reduction for Large-Scale Control Systems**. Mathematical Problems in Engineering 2014:1–13, 2014. DOI 10.1155/2014/843869.

[8] C. Himpe and M. Ohlberger. **A note on the cross Gramian for non-symmetric systems**. System Science and Control Engineering 4(1): 199–208, 2016. DOI 10.1080/21642583.2016.1215273.

Numerical Challenges

Issues:

1. Numerous simulated state and output trajectories.
2. Result is a dense full-order matrix.
3. Memory copies in accumulating operations.

Treatments:

1. In-place operations / Return-Value-Optimization.
2. Approximate low-rank representation.
3. Parallelization.

Low Rank Empirical Cross Gramian

Column-wise representation of the cross Gramian[9]:

$$W_X = \begin{pmatrix} w_{X,1} & w_{X,2} & \dots & w_{X,N} \end{pmatrix} \in \mathbb{R}^{N \times N}$$

Empirical cross Gramian column:

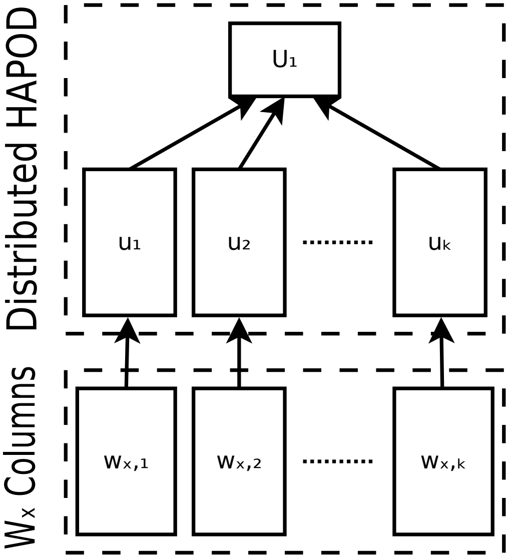
$$w_{X,j} = \frac{1}{M} \sum_{m=1}^{M} \int_0^\infty \psi^{jm}(t) \, \mathrm{d}t \in \mathbb{R}^N$$

$$\psi_i^{jm}(t) = (x_i^m(t) - \bar{x}_i^m)(y_m^j(t) - \bar{y}_m^j) \in \mathbb{R}$$

- This can be done only with the empirical **cross** Gramian!

[9] C. Himpe, T. Leibner and S. Rave. **Hierarchical Approximate Proper Orthogonal Decomposition**. Preprint, arXiv math.NA: 1607.05210, 2016. URL http://arxiv.org/abs/1607.05210.

Distributed Approximate Balancing



Distributed HAPOD

Given a partitioning of a cross Gramian:

$$W_X = \begin{pmatrix} w_{X,1} & \dots & w_{X,K} \end{pmatrix}, \quad w_{X,k} \in \mathbb{R}^{N \times s_k}, \quad \sum_{k=1}^{K} s_k = N$$

Distributed Hierarchical Approximate Proper Orthogonal Decomposition:

$$U_k D_k \overset{\text{POD}}{=} w_{X,k} \quad \text{s.t.} \quad \sqrt{\sum_{i=1}^{s_k} D_{k,ii}^2} < \varepsilon \sqrt{1 - \omega^2}$$

$$\rightarrow U_0 D_0 \overset{\text{POD}}{=} [U_1 D_1, \dots, U_n D_n] \quad \text{s.t.} \quad \sqrt{\sum_{i=1}^{s_k} D_{0,ii}^2} < \varepsilon \omega \frac{N}{\sum_{k=1}^{n} \text{rank}(U_k)}$$

$$\rightarrow \| W_X - U_0 U_0^\intercal W_X \|_2 \leq \varepsilon$$

- An associated error bound for $\|y - y_r\|$ is under construction.

Recipe:

1. Compute distributed empirical cross Gramian.
2. Compute HAPOD modes of cross Gramian partitions.
3. Compute HAPOD from sub-POD modes.
4. Apply approximate balancing.

Properties:

- Requires only basic numerical linear algebra operations.
- Low-communication on distributed memory systems.
- Small memory footprint.

A Procedural Test System

Linear System Structure:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Procedural Modal System[10]:

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_K \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \vdots \\ B_K \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & \dots & C_K \end{pmatrix}$$

$$A_k = \begin{pmatrix} -2\delta_k \omega_k & -\omega_k \\ \omega_k & 0 \end{pmatrix}, \quad B_k = \begin{pmatrix} b_k \\ 0 \end{pmatrix}, \quad C_k = \begin{pmatrix} c_k & \frac{c_k'}{\omega_k} \end{pmatrix}$$

Coefficients:

$$\delta_k > 0, \quad \omega_k > 0$$

[10] W. Gawronski and T. Williams. **Model Reduction for Flexible Space Structures**. Journal of Guidance, 14(1): 68–76, 1991. DOI 10.2541/3.20606.

Model Details

System Dimensions:

- Number of blocks: $K = 1024$
- System dimension: $N = \dim(x(t)) = 2 \cdot 1024 = 2048$
- Input dimension: $M = 1$
- Output dimension: $Q = M = 1$

→ SISO system ⇒ symmetric system

- Parameter: $\delta_k \in \mathcal{U}_{[0, \frac{1}{1000}]}$
- Parameter: $\omega_k \in \mathcal{U}_{[0,100]}$
- Time frame: $[0, 2]$

Empirical Gramians:

- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Linear Cross Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

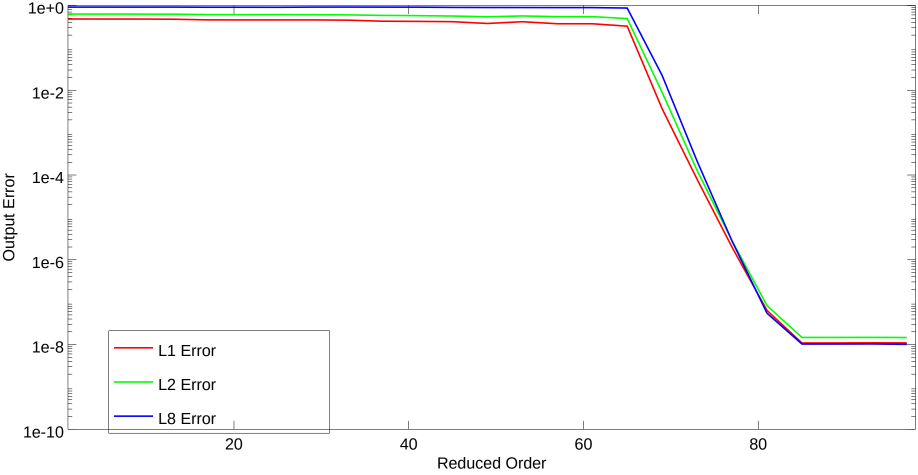
Features:

- Interfaces for: Solver, inner product kernels & distributed memory
- Non-Symmetric option for all cross Gramians
- Compatible with OCTAVE and MATLAB
- Vectorized and parallelizable
- Open-source licensed

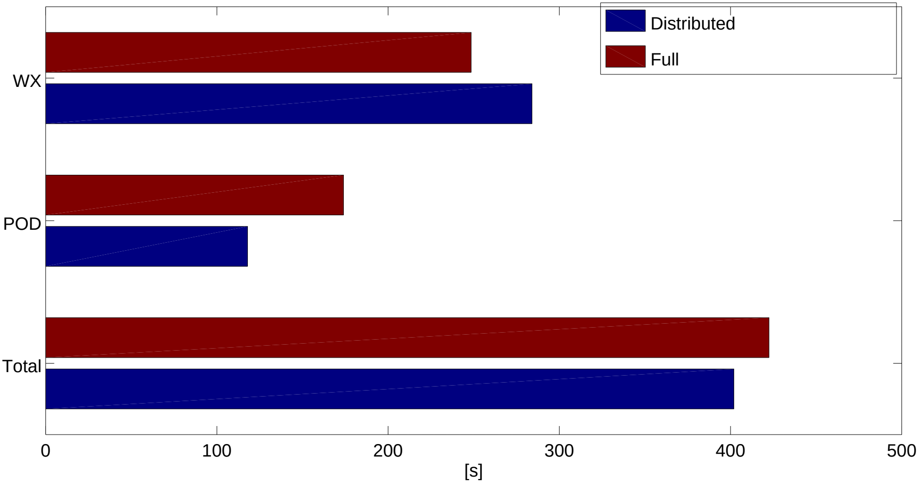
More info: http://gramian.de

C. Himpe. **emgr - The empirical Gramian Framework**. arXiv (cs.MS): 1611.00675, 2016. URL http://arxiv.org/abs/1611.00675.

Numerical Results (Error)



Numerical Results (Timings)



Summary

- Empirical cross Gramian
- Distributed computation
- Recycle partitioning for HAPOD
- SVD-based approximate balancing
- Low communication & memory

More on the **HAPOD**: Thursday, 14:40, C13 HS2, S22: S. Rave

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