MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

[20 YEARS
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An Introduction to Model Reduction

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Mathematical Key Technologies for Evolving Energy Grids

Cooperation Project Partners:

- Fraunhofer SCAI
- Fraunhofer ITWM
- **Max Planck Institute Magdeburg** (Model Reduction)
- Technische Universität Berlin
- Technische Universität Dortmund
- Humboldt Universität zu Berlin
- Friedrich-Alexander Universität Erlangen-Nürnberg
- PSI AG

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Model Reduction in MathEnergy

Complexity Reduction:

- Mathematical Complexity
- Algorithmical Complexity
- Numerical Complexity
- **Dimensional Complexity**

Model:

- Gas Transport Networks
- Transient Simulations
- **Large-Scale Networks**
- Realistic Modeling

Model Reduction:

- Faster Forward Simulation
- **State-Space Reduction**
- Automatical Computation
- Regularising Effect

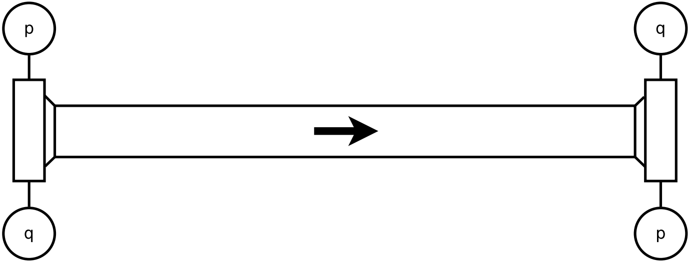
Reduction:

- Input-to-Output Mapping
- Globally Parametric
- Structure-Preserving
- **Nonlinearity**

Input-Output Model: Pipe

Input:

- Pressure @ Inlet (Supply)
- Mass-Flux @ Outlet (Demand)



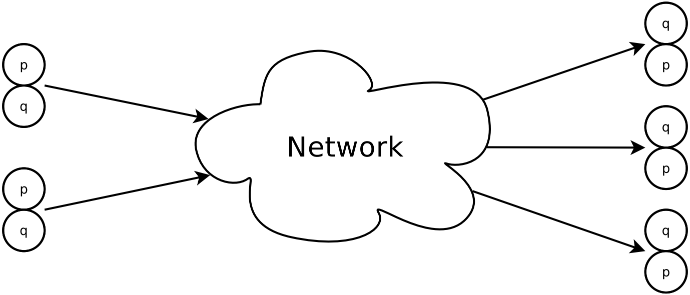
Output:

- Mass-Flux @ Inlet (Supply)
- Pressure @ Outlet (Demand)

Input-Output Model: Network

Input:

- Pressure @ Supply Nodes
- Mass-Flux @ Demand Nodes



Output:

- Mass-Flux @ Supply Nodes
- Pressure @ Demand Nodes

Input-Output Systems

Input-Output Systems:

$$\dot{x}(t) = f(x(t), u(t), \theta)$$
$$y(t) = g(x(t), u(t), \theta)$$

System Components:

- Input: $u : \mathbb{R} \to \mathbb{R}^M$
- State: $x : \mathbb{R} \to \mathbb{R}^N$
- Output: $y : \mathbb{R} \to \mathbb{R}^Q$
- Parameter: $\theta \in \mathbb{R}^P$
- Vector Field: $f : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^N$
- Output Functional: $g : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$

Model Order Reduction (MOR)

Reduced Input-Output Systems:

$$\dot{x}_r(t) = f_r(x_r(t), u(t), \theta)$$
$$\tilde{y}(t) = g_r(x_r(t), u(t), \theta)$$

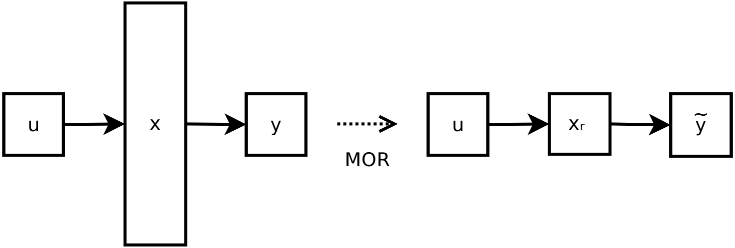
System Components:

- Reduced State: $x_r : \mathbb{R} \to \mathbb{R}^n$
- Reduced Order: $n \ll N$
- Approximate Output: $\tilde{y} : \mathbb{R} \to \mathbb{R}^Q$
- Reduced Vector Field: $f_r : \mathbb{R}^n \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^n$
- Reduced Output Functional: $g_r : \mathbb{R}^n \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$
- Reduced Order Model Quality: $\|y(\theta) - \tilde{y}(\theta)\|_? \ll 1$

Why Does State-Space Reduction Work?

Rationale:

- $\dim(x(t)) = N \gg M = \dim(u(t))$
- $\dim(x(t)) = N \gg Q = \dim(y(t))$



Tools:

- Mathematical System Theory
- State-of-the-Art Numerical Methods

Projection-Based Model Reduction

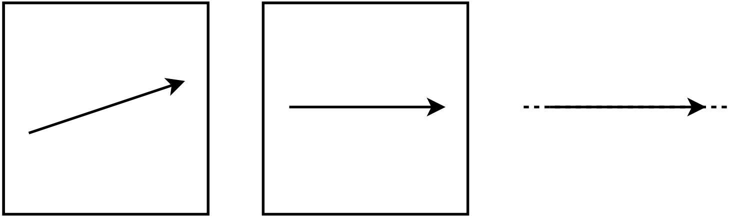
Projected Input-Output Systems:

$$\begin{aligned} \dot{x}_r(t) &= V^\intercal f(Ux_r(t), u(t), \theta) \\ \tilde{y}(t) &= \quad g(Ux_r(t), u(t), \theta) \end{aligned}$$

System Components:

- Reducing Truncated Projection: $V \in \mathbb{R}^{N \times n}$
- Reconstructing Truncated Projection $U \in \mathbb{R}^{N \times n}$
- (Bi-)Orthogonality: $V^\intercal U = \mathbb{1}$
- Reduced State: $x_r := V^\intercal x$
- Reduced Vector Field: $f_r := V^\intercal \circ f \circ U$
- Reduced Output Functional: $g_r := g \circ U$

An Oversimplified Example



1. Original trajectory in phase space.
2. Projection to a simpler space.
3. Truncation of irrelevant dimensions.

Task: Compute U and V

- Input-to-State mapping (controllability)
- State-to-Output mapping (observability)
- Extensive theory for linear systems (Balanced truncation)
- Nonlinear systems hot research topic (Gas Networks are nonlinear)
- Our approach: Data-driven (empirical Gramians)
- Many other methods: POD, RB, H2, QB, DMD, ...
- Most methods are related to the Singular Value Decomposition.

Summary

- Model reduction is **not** magic even though it may appear as such.
- Mathematics reveals model redundancies for practically all models.
- Come to my poster (**27**)! Learn about methods and variants.

http://himpe.science

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