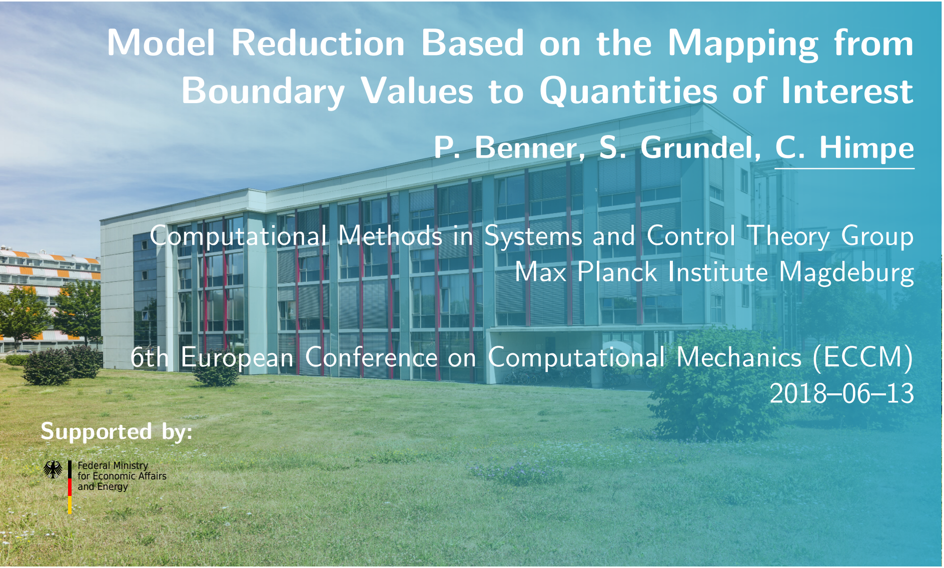
MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

20 YEARS 1998-2018



Model Reduction Based on the Mapping from Boundary Values to Quantities of Interest

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Motivation

The Gas Network Situation:

- Volatile renewable energies.
- Fluctuating supply and demand.
- Fast response of gas-fired plants.
- Day-ahead forecasts.
- Many simulations before dispatch.
- MathEnergy project.

Challenges

- **Multi-Scale**
- **Large-Scale**
- Nonlinear
- Hyperbolic
- Parametric

$\rightarrow$ Structured (Data-Driven) Model Reduction

Multi-Scale Model

Multi-Scale Properties:

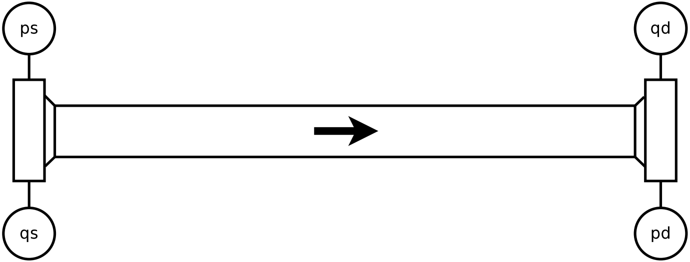
- Coupling of quantites,
- evolving on different scales.
- Practical issues:
- Numerical annihilation,
- Operator condition.

We Assume:

- Spatially discrete (PDAE $\to$ DAE),
- Index reduced (DAE $\to$ ODE),
- Sufficiently regular boundary values (i.e.: L_2).

Simple Example

Gas Flow in a Pipe:



- Modeled by Euler equations,
- coupling mass-flow and pressure.
- For example:
  - Boundary values: p_s, q_d,
  - Quantites of Interest: p_d, q_s.

Semi-Discrete Model

Coupled Input-Output System:

$$\dot{p}(t) = f_p(p(t), q(t), u(t), \theta)$$
$$\dot{q}(t) = f_q(p(t), q(t), u(t), \theta)$$
$$y(t) = \ g(p(t), q(t), u(t), \theta)$$

- Large State: $p : \mathbb{R} \to \mathbb{R}^{N_p}$ (Large Degrees of Freedom)
- Small State: $q : \mathbb{R} \to \mathbb{R}^{N_q}$ (Small Degrees of Freedom)
- Input: $u : \mathbb{R} \to \mathbb{R}^{M}$ (Boundary Values)
- Output: $y : \mathbb{R} \to \mathbb{R}^{Q}$ (Quantites of Interest)
- Parameter: $\theta \in \mathbb{R}^{P}$
- Large Vector Field: $f_p : \mathbb{R}^{N_p} \times \mathbb{R}^{N_q} \times \mathbb{R}^{M} \times \mathbb{R}^{P} \to \mathbb{R}^{N_p}$
- Small Vector Field: $f_q : \mathbb{R}^{N_p} \times \mathbb{R}^{N_q} \times \mathbb{R}^{M} \times \mathbb{R}^{P} \to \mathbb{R}^{N_q}$
- Output Functional: $g : \mathbb{R}^{N_p} \times \mathbb{R}^{N_q} \times \mathbb{R}^{M} \times \mathbb{R}^{P} \to \mathbb{R}^{Q}$

Reducibility

Input-Output Map:

- from Boundary Values u,
- via Degrees of Freedom (p, q),
- to Quantites of Interest y.

$$u \quad \mapsto \quad (p, q) \quad \mapsto \quad y$$

- $N_p = \dim(p(t))$
- $N_q = \dim(q(t))$
- $N_p + N_q \gg 1$
- $M = \dim(u(t)) \ll N_p + N_q$
- $Q = \dim(y(t)) \ll N_p + N_q$

Model Reduction

Reduced Order System:

$$\begin{aligned}
\dot{p}_r(t) &= f_{p,r}(p_r(t), q_r(t), u(t), \theta) \\
\dot{q}_r(t) &= f_{q,r}(p_r(t), q_r(t), u(t), \theta) \\
\tilde{y}(t) &= \quad g_r(p_r(t), q_r(t), u(t), \theta)
\end{aligned}$$

- Reduced Large State: $p_r : \mathbb{R} \to \mathbb{R}^{n_p}$, $n_p \ll N_p$
- Reduced Small State: $q_r : \mathbb{R} \to \mathbb{R}^{n_q}$, $n_q \ll N_q$
- Approximate Output: $\tilde{y} : \mathbb{R} \to \mathbb{R}^Q$
- Reduced Large Vector Field: $f_{p,r} : \mathbb{R}^{n_p} \times \mathbb{R}^{n_q} \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^{n_p}$
- Reduced Small Vector Field: $f_{q,r} : \mathbb{R}^{n_p} \times \mathbb{R}^{n_q} \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^{n_q}$
- Reduced Output Functional: $g_r : \mathbb{R}^{n_p} \times \mathbb{R}^{n_q} \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$
- Reduction Error: $\|y(\theta) - \tilde{y}(\theta)\| \ll 1$

Structured Model Reduction

Structured System:

$$\dot{p}(t) = f_p(p(t), q(t), u(t), \theta)$$
$$\dot{q}(t) = f_q(p(t), q(t), u(t), \theta)$$
$$y(t) = g(p(t), q(t), u(t), \theta)$$

Structured (Truncated) Projections:

$$V_p^\intercal(p) = p_r, \quad U_p(p_r) \approx p, \quad V_q^\intercal(q) = q_r, \quad U_q(q_r) \approx q$$

- Compute separate transformations for large and small equations.
- Per component computation.
- Per component approximation.
- Error estimator for linear systems[1].

[1] H. Sandberg and R.M. Murray. **Model reduction of interconnected linear systems**. Optimal Control Applications and Methods, 30(3): 225–245, 2009.

Projection-Based Model Reduction

Projected System:

$$\begin{aligned}
\dot{p}_r(t) &= V_p^\intercal f_p(U_p p_r(t), U_q q_r(t), u(t), \theta) \\
\dot{q}_r(t) &= V_q^\intercal f_q(U_p p_r(t), U_q q_r(t), u(t), \theta) \\
\tilde{y}(t) &= \qquad g(U_p p_r(t), U_q q_r(t), u(t), \theta)
\end{aligned}$$

- Reducing Large Projection: $V_p \in \mathbb{R}^{N_p \times n_p}$
- Reducing Small Projection: $V_q \in \mathbb{R}^{N_q \times n_q}$
- Reconstructing Large Truncated Projection: $U_p \in \mathbb{R}^{N_p \times n_p}$
- Reconstructing Small Truncated Projection: $U_q \in \mathbb{R}^{N_q \times n_q}$
- Bi-Orthogonality: $V_p^\intercal U_p = \mathbb{1}_{n_p}$, $V_q^\intercal U_q = \mathbb{1}_{n_q}$

$\rightarrow$ **Task**: Compute V_p, U_p, V_q, U_q

Balanced Truncation

Linear Time-Invariant System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Controllability Operator:

$$\mathcal{C}(u)(t) := \int_{-\infty}^{0} \mathrm{e}^{At} \, Bu(-t) \, \mathrm{d}t$$

Observability Operator:

$$\mathcal{O}(x_0)(t) := C \, \mathrm{e}^{At} \, x_0$$

Controllability Gramian:

$$W_C = \int_0^\infty \mathrm{e}^{At} \, BB^\intercal \, \mathrm{e}^{A^\intercal t} \, \mathrm{d}t$$

Observability Gramian:

$$W_O = \int_0^\infty \mathrm{e}^{A^\intercal t} \, C^\intercal C \, \mathrm{e}^{At} \, \mathrm{d}t$$

Balancing (and Truncation):

$$U, V : V^\intercal W_C U = U^{-1} W_O V^{-\intercal} = \mathrm{diag}(\sigma_1, \dots, \sigma_N)$$

Cross Gramian

Conjoining Controllability and Observability:

$$W_X := \int_0^\infty \mathrm{e}^{At}\, BC\, \mathrm{e}^{At}\, \mathrm{d}t$$

Approximate Balancing:

$$W_X \overset{\text{SVD}}{=} UDV$$

- Combines controllability with observability operator.
- Basic variant only for square systems.
- Non-square extensions exist[2]
- Classic or approximate balancing.

[2] C.H. and M. Ohlberger. **A note on the cross Gramian for non-symmetric systems**. System Science and Control Engineering 4(1): 199–208, 2016.

Data-Driven Computation (linear):

$$W_X := \int_0^\infty (\mathrm{e}^{At} B)(\mathrm{e}^{A^\intercal t} C)^\intercal \,\mathrm{d}t$$

- Cross-covariance of primal and adjoint impulse response,
- for linear systems only.

Data-Driven Computation (nonlinear):

$$\widehat{W}_X := \sum_{i=m}^{M} \int_0^\infty \Psi^m(t)\,\mathrm{d}t$$

$$\Psi_{ij}^m = (x_i^m(t) - \bar{x}_i)(y_m^j(t) - \bar{y}_m)$$

- cross-covariance of state and output trajectories,
- for linear systems equivalent to above[3].

[3] C.H. and M. Ohlberger. **Cross-Gramian-Based Combined State and Parameter Reduction for Large-Scale Control Systems**. Mathematical Problems in Engineering, 2014: 1–13, 2014.

Empirical Structured Cross Gramian

Large Empirical Cross Gramian:

$$\widehat{W}_{X,p} := \sum_{i=m}^{M} \int_0^\infty \Psi_p^m(t)\,\mathrm{d}t$$

$$\Psi_{p,ij}^m = (p_i^m(t) - \bar{p}_i)(y_m^j(t) - \bar{y}_m)$$

$$\widehat{W}_{X,p} \overset{\text{SVD}}{=} U_p D_p V_p^\intercal$$

Small Empirical Cross Gramian:

$$\widehat{W}_{X,q} := \sum_{i=m}^{M} \int_0^\infty \Psi_q^m(t)\,\mathrm{d}t$$

$$\Psi_{q,ij}^m = (q_i^m(t) - \bar{q}_i)(y_m^j(t) - \bar{y}_m)$$

$$\widehat{W}_{X,q} \overset{\text{SVD}}{=} U_q D_q V_q^\intercal$$

Note: Output scaling important!

Pipe Model

Isothermal Euler Equations[4]:

$$\frac{\partial}{\partial t}\rho = -\frac{1}{S}\frac{\partial}{\partial x}q$$

$$\frac{\partial}{\partial t}q = -S\frac{\partial}{\partial x}p - Sg\rho\frac{\partial}{\partial x}h - \frac{\lambda}{2DS}\frac{q|q|}{\rho}$$

$$p = R_S T_0 z \rho$$

- Density: $\rho(x,t)$
- Mass-Flux: $q(x,t)$
- Pressure: $p(x,t)$
- Elevation: $h(x)$
- Constants: S, g, D
- Parameters: T_0, R_S
- Friction Factor: $\lambda(q)$
- Compressibility Factor: $z(p,T)$

[4] P. Benner, S. Grundel, C.H., C. Huck, T. Streubel, C. Tischendorf. **Gas Network Benchmark Models**. In: Differential Algebraic Equation Forum (Accepted), 2018.

Network Model

Network as a Graph $(\mathcal{N}, \mathcal{E})$:

- Node set $\mathcal{N}$ (Junctions)
- Edge set $\mathcal{E}$ (Pipes)

Node Types:

- Internal nodes $\mathcal{N}_0$
- Supply nodes $\mathcal{N}_s$
- Demand nodes $\mathcal{N}_d$

Kirchhoff Rules[5]:

1. $p_i = p_s, \quad p_i \in \mathcal{N}_s$
2. $\sum_{j \in \mathcal{E}_-} q_j - \sum_{k \in \mathcal{E}_+} q_k = q_d, \quad q_d \in \mathcal{N}_d$

$\rightarrow$ Partial Differential Algebraic Equation (PDAE)

[5] T.P. Azevedo-Perdicoúlis and G. Jank. **Modelling Aspects of Describing a Gas Network Through a DAE System**. 3rd IFAC Symposium on System Structure and Control 40(20): 40–45, 2007.

Input-Output Model

Semi-Discrete Gas Network Model:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & A_{pq} \\ A_{qp} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 & B_d \\ B_s & 0 \end{pmatrix} \begin{pmatrix} p_s \\ q_d \end{pmatrix} + \begin{pmatrix} 0 \\ f_q(p, q, p_s, q_d, \theta) \end{pmatrix}$$

$$\begin{pmatrix} p_d \\ q_s \end{pmatrix} = \begin{pmatrix} C_d & 0 \\ 0 & C_s \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

- Multi-Scale ($\sim 10^5$) $\rightarrow$ Structured Reduction
- Always square $\rightarrow$ Cross Gramian
- Nonlinear (Friction & Compressibility) $\rightarrow$ Empirical Gramian
- Stiff "linear" part $\rightarrow$ IMEX[6] Solver
- 2-dim parameter-space (Temperature & Specific Gas Constant)

[6] S. Grundel, L. Jansen. **Efficient Simulation of Transient Gas Networks Using IMEX Integration Schemes and MOR Methods**. IEEE 54th Annual Conference on Decision and Control: 4579–4584, 2015.

Model Reduction Workflow

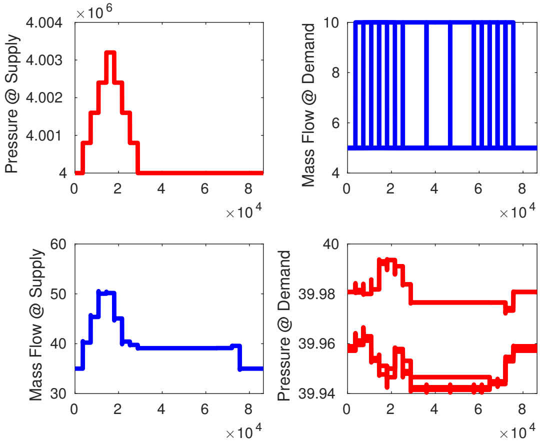
Offline:

1. Design input perturbation
2. Design parameter samples
3. Compute structured empirical Gramians
4. for short time horizons ($< 1\,\mathrm{h}$)
5. Decompose empirical Gramians

Online:

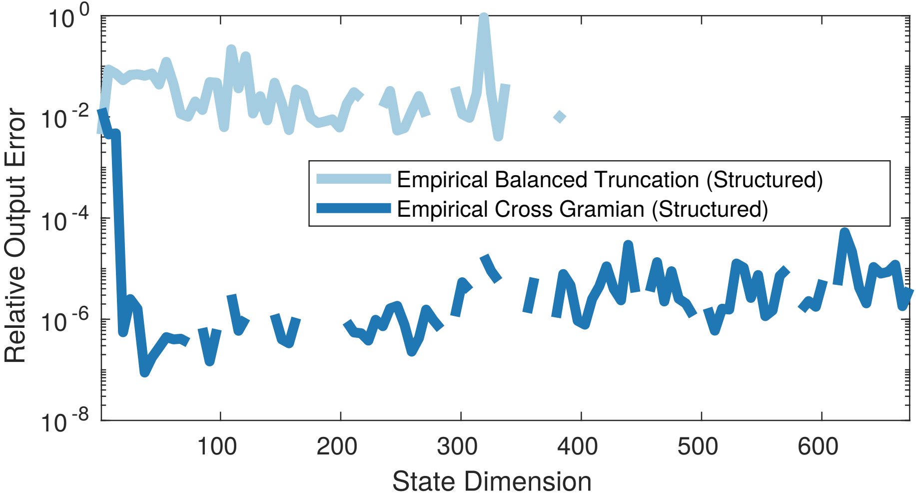
1. Design scenarios
2. Simulate scenarios
3. for long time horizons ($24\,\mathrm{h}$)

Test Network and Scenario



- Distribution network
- Tree topology
- Single supply, multiple demands
- About 700 DoFs
- $24\,\mathrm{h}$ time horizon
- $1\,\mathrm{min}$ time resolution

Numerical Results



(Unstructured or Structured) POD, (Unstructured Empircal BT & WX), DMD-Galerkin, (Unstructured or Structured) Linearized BT & WX did not work!

Summary

- System-theoretic interpretation
- Boundary-Value to Quantity-of-Interest mapping
- Structured empirical cross Gramian
- Output scaling necessary
- Gas network application

http://himpe.science

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