Linear Model Reduction for Nonlinear Input-Output Systems

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5th EUropean Conference on Computational Optimization (EUCCO)
Model order reduction and low-rank approximation for nonlinear problems
2018–09–12

Supported by:

Federal Ministry
for Economic Affairs
and Energy

Motivation

Why:

- Computationally less involved.
- Less accurate but universal.
- Nonlinearity may not be main challenge.
- Nonlinear methods may not apply.
- Theoretical results for linear systems.

Input-Output System

Nonlinear Input-Output System:

$$\dot{x}(t) = f(t, x(t), u(t), \theta)$$
$$y(t) = g(t, x(t), u(t), \theta)$$

- Input: $u : \mathbb{R} \to \mathbb{R}^M$
- State: $x : \mathbb{R} \to \mathbb{R}^N$
- Output $y : \mathbb{R} \to \mathbb{R}^Q$
- Parameter: $\theta \in \mathbb{R}^P$
- Vector Field: $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^N$
- Output Functional: $g : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$

Reduced Order Model (ROM):

$$\dot{x}_r(t) = f_r(t, x_r(t), u(t), \theta)$$
$$\tilde{y}(t) = g_r(t, x_r(t), u(t), \theta)$$

- Reduced State: $x_r : \mathbb{R} \to \mathbb{R}^n$, $n \ll N$
- Reduced Vector Field: $f_r : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^n$
- Reduced Output Func.: $g_r : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$
- Approximate Output: $\tilde{y} : \mathbb{R} \to \mathbb{R}^Q$
- Goal I: $\|y(\theta) - \tilde{y}(\theta)\| \ll 1$
- Goal II: Preserve stability, structure, etc.

Projected System:

$$\begin{aligned} \dot{x}_r(t) &= V^\intercal f(t, U x_r(t), u(t), \theta) \\ \tilde{y}(t) &= \quad g(t, U x_r(t), u(t), \theta) \end{aligned}$$

- Reducing projection: $V \in \mathbb{R}^{N \times n}$
- Reconstructing projection: $U \in \mathbb{R}^{N \times n}$
- Projection ansatz: $x_r(t) := V^\intercal x(t) \to x(t) \approx U x_r(t)$
- (Bi-)Orthogonality: $V^\intercal U = \mathbb{1}$
- Galerkin projection: $V = U$
- Petrov-Galerkin projection: $U \neq V$

Linear Model Reduction

Linear Time-Invariant System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Reduced Linear System:

$$\dot{x}_r(t) = V^\intercal(A(Ux_r(t))) + Bu(t))$$
$$\tilde{y}(t) = C(Ux_r(t))$$

- System matrix: $A \in \mathbb{R}^{N \times N}$
- Input matrix: $B \in \mathbb{R}^{N \times M}$
- Output matrix: $C \in \mathbb{R}^{Q \times N}$

Linear Model Reduction

Linear Time-Invariant System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Reduced Linear System:

$$\dot{x}_r(t) = (V^\intercal AU)x_r(t) + (V^\intercal B)u(t)$$
$$\tilde{y}(t) = (CU)x_r(t)$$

- System matrix: $A \in \mathbb{R}^{N \times N}$
- Input matrix: $B \in \mathbb{R}^{N \times M}$
- Output matrix: $C \in \mathbb{R}^{Q \times N}$

Linear Model Reduction

Linear Time-Invariant System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Reduced Linear System:

$$\dot{x}_r(t) = A_r x_r(t) + B_r u(t)$$
$$\tilde{y}(t) = C_r x_r(t)$$

- Reduced System Matrix: $A_r \in \mathbb{R}^{n \times n}$
- Reduced Input Matrix: $B_r \in \mathbb{R}^{n \times M}$
- Reduced Output Matrix: $C_r \in \mathbb{R}^{Q \times n}$

Linear Model Reduction

Nonlinear System:

$$\dot{x}(t) = f(t, x_r(t), u(t), \theta)$$
$$y(t) = g(t, x_r(t), u(t), \theta)$$

Reduced Nonlinear System:

$$\dot{x}_r(t) = V^\intercal f(t, U x_r(t), u(t), \theta)$$
$$\tilde{y}(t) = \quad g(t, U x_r(t), u(t), \theta)$$

- Reduced Vector Field: $f_r : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^n$
- Reduced Output Func.: $g_r : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$
- Linear model reduction for a nonlinear system.

Briefly: Hyper Reduction

Tackling the lifting bottleneck:

- Reducing the reduced order model.
- Approximate action of projected nonlinearity.
- Solely low-dimensional operator applications.

Hyper Reduction Methods:

- Discrete Empirical Interpolation Method[1] (DEIM)
- Dynamic Mode Decomposition[2] (DMD)
- Numerical Linearization[3]
- ...

[1] S. Chaturantabut, D.C. Sorensen. **Nonlinear model reduction via discrete empirical interpolation**. SIAM Journal on Scientific Computing, 32: 2737–2764, 2010. doi:10.1137/090766498.

[2] M.O. Williams, P.J. Schmid, J.N. Kutz. **Hybrid Reduced-Order Integration with Proper Orthogonal Decomposition and Dynamic Mode Decomposition**. Multiscale Modeling & Simulation, 11: 522–544, 2013. doi:10.1137/120874539.

[3] B.C. Moore. **Principal Component Analysis in Nonlinear Systems: Preliminary Results**. 18th IEEE Conference on Decision and Control including the Symposium on Adaptive Processes, 2: 1057–1060, 1979. doi:10.1109/CDC.1979.270114.

Superlinear Model Reduction

Idea:

- Use nonlinear information to build linear subspaces.
- Simulate perturbed systems to capture nonlinearity.
- Data-driven, hence specific to operating region.
- Independent of system structure.
- Superior to linearization[4].

[4] I. Dones, S. Skogestad, H.A. Preisig. **Application of Balanced Truncation to Nonlinear Systems**. Industrial & Engineering Chemistry Research, 50: 10093–10101, 2011.

POD[5] (Simplified):

$$X^k = \begin{pmatrix} x_1^k & \dots & x_T^k \end{pmatrix} \in \mathbb{R}^{N \times T}$$
$$\mathcal{X} := \begin{bmatrix} X^1 & \dots & X^K \end{bmatrix}$$
$$\mathcal{X} \overset{\text{tSVD}}{=} UDW$$
$$V := U$$

- State trajectory X^k,
- for k-th perturbation,
- with T snapshots each.

[5] L. Sirovich. **Turbulence and the Dynamics of Coherent Structures: I-Coherent structures, II-Symmetries and transformations, III-Dynamics and scaling**. Quarterly of Applied Mathematics, 45(3): 561–590, 1987.

Extended DMD-Galerkin[6] (Simplified):

$$X^k = \begin{pmatrix} x_1^k & \dots & x_T^k \end{pmatrix} \in \mathbb{R}^{N \times T}$$

$$\to X_0^k := \begin{pmatrix} x_1^k & \dots & x_{T-1}^k \end{pmatrix}, X_1^k := \begin{pmatrix} x_2^k & \dots & x_T^k \end{pmatrix}$$

$$\to \mathcal{X}_0 := \begin{bmatrix} X_0^1 & \dots X_0^K \end{bmatrix}, \mathcal{X}_1 := \begin{bmatrix} X_1^1 & \dots X_1^K \end{bmatrix}$$

$$\mathcal{X}_1 \stackrel{!}{=} A \mathcal{X}_0 \to A \approx \mathcal{X}_1 \mathcal{X}_0^+$$

$$\mathcal{X}_1 \mathcal{X}_0^+ T = \Lambda T$$

$$U := \mathrm{orth}(T), \quad V := U$$

[6] A. Alla and J.N. Kutz. **Nonlinear Model Order Reduction via Dynamic Mode Decomposition**. SIAM J Sci Comput, 39(5): B778–B796, 2017.

Empirical Gramians[7] (Simplified):

$$X^{mk} = \begin{pmatrix} x_1^{mk} & \dots & x_T^{mk} \end{pmatrix} \to W_C := \sum_{m,k} \alpha^{mk} X^{mk} (X^{mk})^\intercal,$$

$$Y^{q\ell} = \begin{pmatrix} y_q^{\ell 1} & \dots & y_q^{\ell N} \end{pmatrix} \to W_O := \sum_{q,\ell} \beta^{n\ell} (Y^{q\ell})^\intercal Y^{q\ell}$$

$$W_C^{\frac{1}{2}} W_O^{\frac{1}{2}} \overset{\text{tSVD}}{=} UDV$$

- q-th component of output trajectory: $y_q^{\ell n}$,
- for ℓ-th perturbation,
- of the n-th initial state component.

[7] S. Lall, J.E. Marsden, and S. Glavaski. **Empirical Model Reduction of Controlled Nonlinear Systems**. In: Proceedings of the 14th IFAC Congress, F: 473–478, 1999.

Empirical Cross Gramian[8] (Simplified):

$$X^{mk} := \begin{pmatrix} x_1^{mk} & \dots & x_T^{mk} \end{pmatrix}$$

$$Y^{\ell m} := \begin{pmatrix} y_m^{\ell 1} & \dots & y_m^{\ell N} \end{pmatrix}$$

$$W_X := \sum_{k,\ell,m} \gamma^{k\ell m} X^{mk} Y^{\ell m}$$

$$W_X W_X \overset{\text{tSVD}}{=} UDV$$

[8] C.H. and M. Ohlberger. **Cross-Gramian-Based Model Reduction: A Comparison**. In: Modeling, Simulation and Applications, 17: 271–283, 2017.

Comparison

Model Reduction Methods:

- Proper Orthogonal Decomposition
- Dynamic Mode Decomposition-Galerkin
- Empirical Balanced Truncation
- Empirical Approximate Balancing

Preliminary Results

Isothermal Euler Equations[9]:

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho &= -\frac{1}{S}\frac{\partial}{\partial x}q \\
\frac{\partial}{\partial t}q &= -S\frac{\partial}{\partial x}p - Sg\rho\frac{\partial}{\partial x}h - \frac{\lambda}{2DS}\frac{q|q|}{\rho} \\
p &= R_S T_0 z \rho
\end{aligned}
$$

- Density: $\rho(x,t)$
- Mass-Flux: $q(x,t)$
- Pressure: $p(x,t)$
- Elevation: $h(x)$
- Constants: S, g, D
- Parameters: T_0, R_S
- Friction Factor: $\lambda(q)$
- Compressibility Factor: $z(p,T)$

[9] P. Benner, S. Grundel, C.H., C. Huck, T. Streubel, C. Tischendorf. **Gas Network Benchmark Models**. In: Differential Algebraic Equation Forum (Online First), 2018.

Test System

Semi-Discrete Gas Network Model[10]:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & A_{pq} \\ A_{qp} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 & B_d \\ B_s & 0 \end{pmatrix} \begin{pmatrix} p_s \\ q_d \end{pmatrix} + \begin{pmatrix} 0 \\ f_{\mathsf{fr}}(p, q, p_s, q_d, \theta) \end{pmatrix}$$

$$\begin{pmatrix} p_d \\ q_s \end{pmatrix} = \begin{pmatrix} C_d & 0 \\ 0 & C_s \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

Pipe network preprocessing:

- Index reduction (Analytic).
- Spatial discretization (Midpoint).
- Pressure and mass-flux states.
- IMEX integrator.

[10] S. Grundel, L. Jansen, N. Hornung, T. Clees, C. Tischendorf, P. Benner. **Model Order Reduction of Differential Algebraic Equations Arising from the Simulation of Gas Transport Networks**. In: Progress in Differential-Algebraic Equations, Differential-Algebraic Equations Forum: 183–205, 2014.

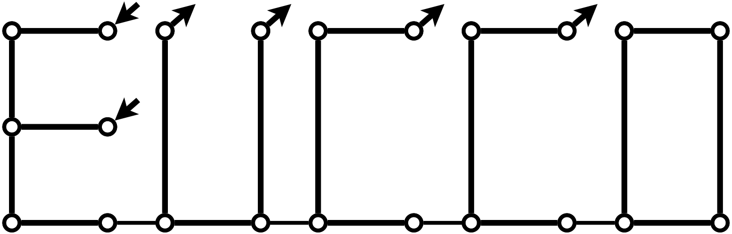
Structured Reduced Order Model[11]:

$$\begin{aligned}\dot{p}_r(t) &= V_p^\intercal f_p(t, U_p p_r(t), U_q q_r(t), u(t), \theta) \\ \dot{q}_r(t) &= V_q^\intercal f_q(t, U_p p_r(t), U_q q_r(t), u(t), \theta) \\ \tilde{y}(t) &= \qquad g(t, U_p p_r(t), U_q q_r(t), u(t), \theta)\end{aligned}$$

- Reducing pressure projection: $V_p \in \mathbb{R}^{N_p \times n_p}$
- Reducing mass-flux projection: $V_q \in \mathbb{R}^{N_q \times n_q}$
- Reconstructing pressure projection: $U_p \in \mathbb{R}^{N_p \times n_p}$
- Reconstructing mass-flux projection: $U_q \in \mathbb{R}^{N_q \times n_q}$
- Bi-Orthogonality: $V_p^\intercal U_p = \mathbb{1}_{n_p}$, $V_q^\intercal U_q = \mathbb{1}_{n_q}$

[11] H. Sandberg and R.M. Murray. **Model reduction of interconnected linear systems**. Optimal Control Applications and Methods, 30(3): 225–245, 2009.

Test Network



Network Properties:

- Nodes: 22
- Edges: 22
- Supplies: 2
- Demands: 4

System Dimensions:

- Pressure states: 518
- Mass-flux states: 520
- Inputs: 6
- Outputs: 6

Model Reduction Workflow

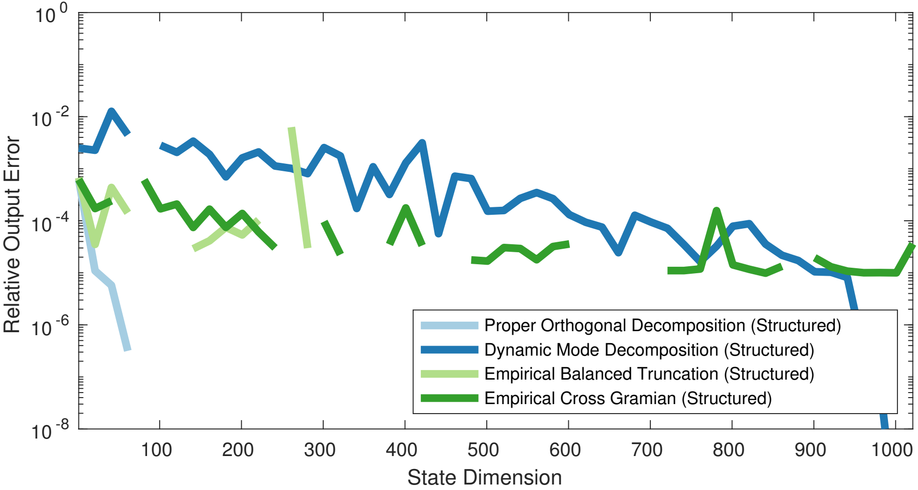
Offline:

1. Design input and parameter samples.
2. Compute assciated trajectories,
3. for short time horizons ($< 1\,\mathrm{h}$).
4. Compute projections from trajectories.

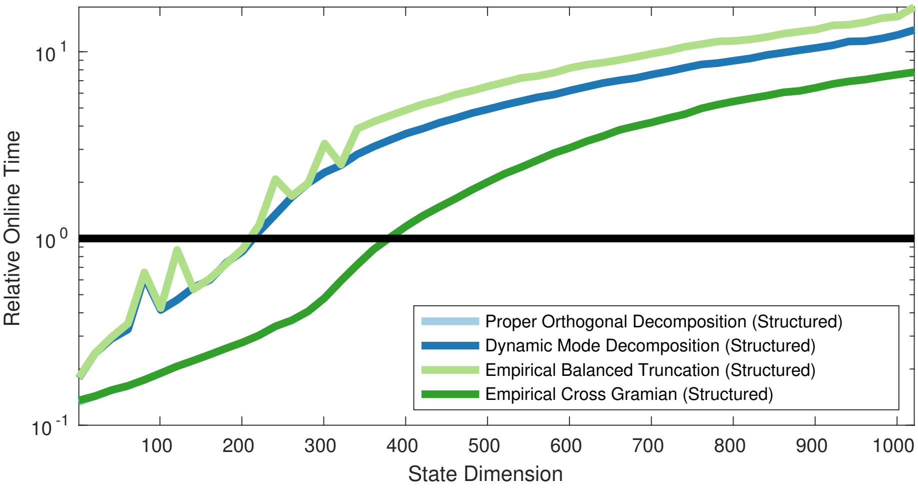
Online:

1. Design scenarios.
2. Simulate scenarios,
3. for long time horizons ($24\,\mathrm{h}$).

Model Reduction Error



Online Time



Relative Online Time
State Dimension
10^{1}
10^{0}
10^{-1}
100
200
300
400
500
600
700
800
900
1000
Proper Orthogonal Decomposition (Structured)
Dynamic Mode Decomposition (Structured)
Empirical Balanced Truncation (Structured)
Empirical Cross Gramian (Structured)

Next Steps

Before to Hyper Reduction:

- Comparison: Benchmarks[12]
- Preprocessing: A-Priori Error Indicator (WIP)
- Parallelization: HAPOD[13]
- Postprocessing: Stabilization[14]
- ROM Efficiency: Memory vs Compute (WIP)

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[12] U. Baur, P. Benner, B. Haasdonk, C.H., I. Martini and M. Ohlberger. **Comparison of Methods for Parametric Model Order Reduction of Time-Dependent Problems**. In: Model Reduction and Approximation: Theory and Algorithms, SIAM: 377–407, 2017.

[13] C.H. and T. Leibner and S. Rave. **Hierarchical Approximate Proper Orthogonal Decomposition**. SIAM Journal on Scientific Computing: Accepted, 2018.

[14] P. Benner, C.H., T. Mitchell. **On Reduced Input-Output Dynamic Mode Decomposition**. Advances in Computational Mathematics: 1–18, 2018.

Summary

Take-home message:

- Nonlinear information for linear subspace.
- Quality of linear subspace.
- Online time may vary.

http://himpe.science

Acknowledgment:
Supported by the German Federal Ministry for Economic Affairs and Energy, in the joint project: "**MathEnergy** – Mathematical Key Technologies for Evolving Energy Grids", sub-project: Model Order Reduction (Grant number: 0324019**B**).