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CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

On “A Koopman Operator Approach for Computing and Balancing Gramians for Discrete Time Nonlinear Systems” by: E. Young, Z. Liu and N.O. Hodas

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CSC

Outline

1. The Koopman Operator
2. Koopman Observability Gramian
3. Koopman Controllability Gramian
4. Koopman Balanced Truncation

Koopman Operator

Nonlinear Difference Equation:

$$x_{k+1} = f(x_k)$$

- State: $x : \mathbb{N} \to \mathbb{R}^N$
- Transition Function: $f : \mathbb{R}^N \to \mathbb{R}^N$

Observable $\psi : \mathbb{R}^N \to \mathbb{R}^P$:

$$z_k := \psi(x_k)$$

Koopman Operator $\mathcal{K}$:

$$\psi(x_{k+1}) = \psi(f(x_k)) = \mathcal{K}\psi(x_k)$$

Koopman Properties

- $f \in C^\infty \Rightarrow \exists \mathcal{K}$.
- $\mathcal{K}$ evolves the **observable** trajectory in time.
- $\mathcal{K}$ is linear (!)
- $\mathcal{K}$ is (countably) infinite-dimensional.

Koopman Approximation

Given State Trajectory Data:

$$X = [x_1, x_2, \dots, x_T] \in \mathbb{R}^{N \times T}$$

Partition as:

$$X_0 := [x_1, \dots, x_{T-1}] \in \mathbb{R}^{N \times (T-1)}, \quad X_1 := [x_2, \dots, x_T] \in \mathbb{R}^{N \times (T-1)}$$

Koopman Operator Approximation:

$$\begin{aligned} \psi(x_{k+1}) &= \mathcal{K}\psi(x_k) \\ \Rightarrow \mathcal{K} &\approx \operatorname*{argmin}_K \|\Psi(X_1) - K\Psi(X_0)\| + \lambda\|K\| \\ \Psi(X) &:= \begin{pmatrix} \psi(X_{*,1}) & \dots & \psi(X_{*,\text{end}}) \end{pmatrix} \end{aligned}$$

For $\|\cdot\|_{\text{Fro}}$ this is Dynamic Mode Decomposition (DMD).

Discrete Control-Affine Input-Output System:

$$\begin{aligned} x_{k+1} &= f(x_k) + g(x_k)u_k \\ y_k &= h(x_k) \end{aligned}$$

- Discrete Time: $k \in \mathbb{N}$
- Input: $u_k : \mathbb{N} \to \mathbb{R}^M$
- State: $x_k : \mathbb{N} \to \mathbb{R}^N$
- Output: $y_k : \mathbb{N} \to \mathbb{R}^Q$
- State Transition Function: $f : \mathbb{R}^N \to \mathbb{R}^N$
- Input Transition Function*: $g : \mathbb{R}^N \to \mathbb{R}^N$
- Output Functional: $h : \mathbb{R}^N \to \mathbb{R}^Q$

Observables

Example Observable:

$$\psi_0(x_k) := \begin{pmatrix} x_k \\ x_k^2 \end{pmatrix}$$

Assumption:

$$y_k \in \operatorname{span}\{\psi(x_k) : k \in \mathbb{N}\} \Rightarrow \exists L_h \in \mathbb{R}^{Q \times P} : y_k = L_h \psi(x_k)$$

Note:

- The observable is generally not the output functional $\psi \neq h$!

Koopman Observability Gramian

Initial-State-to-Output Mapping:

$$y_k = L_h \psi(x_k) = L_h \mathcal{K} \psi(x_{k-1}) = \dots = L_h \mathcal{K}^k \psi(x_0)$$

Koopman Observability Operator:

$$\mathcal{O}_{\mathcal{K}}(x)(k) := L_h \mathcal{K}^k x \Rightarrow \mathcal{O}_{\mathcal{K}}^*(z)(k) = \sum_{l=0}^{k} (L_h \mathcal{K}^l)^* z_l$$

Koopman Observability Gramian:

$$\widetilde{X}_O := \mathcal{O}_{\mathcal{K}}^* \mathcal{O}_{\mathcal{K}} = \sum_{l=0}^{\infty} (L_h \mathcal{K}^l)^* L_h \mathcal{K}^l \in \mathbb{R}^{P \times P}$$

Projected Koopman Observability Gramian:

$$X_O := P_O \widetilde{X}_O P_O^\intercal \in \mathbb{R}^{N \times N}, \quad P_O : M_O \subset \mathbb{R}^P \to \mathbb{R}^N$$

Example: Linear System

Discrete (Finite-Dimensional) Linear System:

$$x_{k+1} = Ax_k$$
$$y_k = Cx_k$$

Koopman Operator, Observable, and Adapter:

$$\mathcal{K} = A$$
$$\psi = \mathbb{1}_N$$
$$L_h = C$$
$$P_O = \mathbb{1}_N$$

Associated Koopman Observability Gramian:

$$X_O = \sum_{l=0}^{\infty} (CA^l)^\intercal CA^l = W_O,$$

which is equal to the common discrete observability Gramian!

Observable $\hat{\psi} : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^P$:

$$z_k := \hat{\psi}(x_k, u_k)$$

Koopman Operator $\mathcal{K}$:

$$\hat{\psi}(x_{k+1}, u_{k+1}) = \hat{\mathcal{K}}\hat{\psi}(x_k, u_k)$$

Assumption & Lemma 4:

$$\hat{\psi}(x_{k+1}, u_{k+1}) = \hat{\psi}(x_{k+1}, 0) = \hat{\mathcal{K}}\hat{\psi}(x_k, u_k)$$
$$\Rightarrow \hat{\psi}(x_{k+1}, u_{k+1}) = \mathcal{K}\psi(x_k) + \mathcal{K}_u\psi_u(u_k)$$
$$\Rightarrow \psi(x_{k+1}) = \hat{\mathcal{K}}\hat{\psi}(x_k, u_k)$$

Koopman Controllability Gramian

Input-to-State Mapping:

$$x_k = \sum_{l=0}^{k} \mathcal{K}^l \mathcal{K}_u u_l$$

Koopman Controllability Operator:

$$\mathcal{C}_\mathcal{K}(u)(k) := \sum_{l=0}^{k} \mathcal{K}^{k-l} \mathcal{K}_u u_l \Rightarrow \mathcal{C}_\mathcal{K}^*(x)(k) = (\mathcal{K}^k \mathcal{K}_u)^* x$$

Koopman Controllability Gramian:

$$\widetilde{X}_C := \mathcal{C}_\mathcal{K} \mathcal{C}_\mathcal{K}^* = \sum_{l=0}^{\infty} \mathcal{K}^l \mathcal{K}_u (\mathcal{K}^l \mathcal{K}_u)^* \in \mathbb{R}^{P \times P}$$

Projected Koopman Controllability Gramian:

$$X_C := P_C \widetilde{X}_C P_C^\intercal \in \mathbb{R}^{N \times N}, \quad P_C : M_C \subset \mathbb{R}^P \to \mathbb{R}^N$$

Example: Linear System

Discrete (Finite-Dimensional) Linear System:

$$x_{k+1} = Ax_k + Bu_k$$

Koopman Operator, Input Koopman Operator, Observable:

$$\begin{aligned} \mathcal{K} &= A \\ \mathcal{K}_u &= B \\ \psi &= \mathbb{1}_N \\ P_C &= \mathbb{1}_N \end{aligned}$$

Associated Koopman Controllability Gramians:

$$X_C = \sum_{l=0}^{\infty} A^l B (A^l B)^\intercal = W_C,$$

which is equal to the common discrete controllability Gramian!

Projection-Based Model Reduction

Given bi-orthogonal truncated projections:

$$U_1 \in \mathbb{R}^{N \times n},\ V_1 \in \mathbb{R}^{n \times N},\ V_1 U_1 = \mathbb{1}$$

Linear Reduced Order Model:

$$\begin{aligned} x_{r,k+1} &= (V_1 A U_1) x_{r,k} + (V_1 B) u_k \\ \tilde{y}_k &= \quad (C U_1) x_{r,k} \end{aligned}$$

Nonlinear Reduced Order Model:

$$\begin{aligned} x_{r,k+1} &= V_1 f(U_1 x_{r,k}, u_k) \\ \tilde{y}_k &= \quad g(U_1 x_{r,k}) \end{aligned}$$

Reminder: Discrete Balanced Truncation

Discrete Controllability Operator & Discrete Observability Operator:

$$\mathcal{C}(u)(k) := \sum_{l=0}^{k} A^{k-l} B u_k, \quad \mathcal{O}(x)(k) := C A^k x$$

Discrete Controllability Gramian & Discrete Observabilty Gramian:

$$W_C := \mathcal{C}\mathcal{C}^* \geq 0, \quad W_O := \mathcal{O}^*\mathcal{O} \geq 0$$

Balancing Transformation & Truncation:

$$W_C^{\frac{1}{2}} W_O W_C^{\frac{1}{2}} \overset{\text{SVD}}{=} T D T^*$$

$$\rightarrow U = W_C^{\frac{1}{2}} T D^{-\frac{1}{2}}, \quad V = D^{\frac{1}{2}} T^* W_C^{-\frac{1}{2}}$$

$$\rightarrow U_1 := U_{*,1\dots n}, \quad V_1 := V_{1\dots n,*}, \quad D_{j,j} > D_{j+1,j+1}$$

Koopman Balanced Truncation

Koopman Controllability & Observability Operator:

$$\mathcal{C}_{\mathcal{K}}(u)(k) := \sum_{l=0}^{k} \mathcal{K}^{k-l} \mathcal{K}_u u_l, \quad \mathcal{O}_{\mathcal{K}}(x)(k) := L_h \mathcal{K}^k x$$

Projected Koopman Controllability & Observability Gramian:

$$X_C := P \mathcal{C}_{\mathcal{K}} \mathcal{C}_{\mathcal{K}}^* P^\intercal \overset{\text{Lemma 5}}{\geq} 0, \quad X_O := P \mathcal{O}_{\mathcal{K}}^* \mathcal{O}_{\mathcal{K}} P^\intercal \overset{\text{Lemma 2}}{\geq} 0$$

Balancing Transformation & Truncation:

$$X_C^{\frac{1}{2}} X_O X_C^{\frac{1}{2}} \overset{\text{SVD}}{=} T D T^*$$

$$\rightarrow U = X_C^{\frac{1}{2}} T D^{-\frac{1}{2}}, \quad V = D^{\frac{1}{2}} T^* X_C^{-\frac{1}{2}}$$

$$\rightarrow U_1 := U_{*,1\dots n}, \;\; V_1 := V_{1\dots n,*}, \;\; D_{j,j} > D_{j+1,j+1}$$

Koopman Singular Values

(Classic) Discrete Hankel Operator & Global A-Priori Error Bound:

$$H := \mathcal{O}\mathcal{C}, \quad \sigma_{n+1}(H) \leq \|G - G_r\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=n+1}^{N} \sigma_i(H)$$

Koopman "Hankel" Operator:

$$H_\mathcal{K} := \mathcal{O}_\mathcal{K}\mathcal{C}_\mathcal{K}$$

Balanced-Truncation-like Error Indicator:

$$\sigma_{n+1}(H_\mathcal{K}) \lesssim \|G - G_r\|_{\mathcal{H}_\infty} \lesssim 2 \sum_{i=n+1}^{N} \sigma_i(H_\mathcal{K})$$

For linear systems the classic error bound emerges!
Also, a linear(ized) realization, i.e.: $\Sigma(\mathcal{K}, \mathcal{K}_u, L_h)$, can be used.

Summary

- Koopman Gramians are a generalization:
  linear systems reduce to linear Gramians.
- How to obtain L_h generally?
- How to obtain P_O, P_C generally?
- Can this be extended to $x_{k+1} = f(x_k, u_k)$?
- Are Koopman Gramians actually
  discrete empirical Gramians with kernels?

E. Yeung, Z. Liu, N.O. Hodas. **A Koopman Operator Approach for Computing and Balancing Gramians for Discrete Time Nonlinear Systems**. arXiv, cs.SY: 1709.08712, 2017. `http://arxiv.org/pdf/1709.08712.pdf`