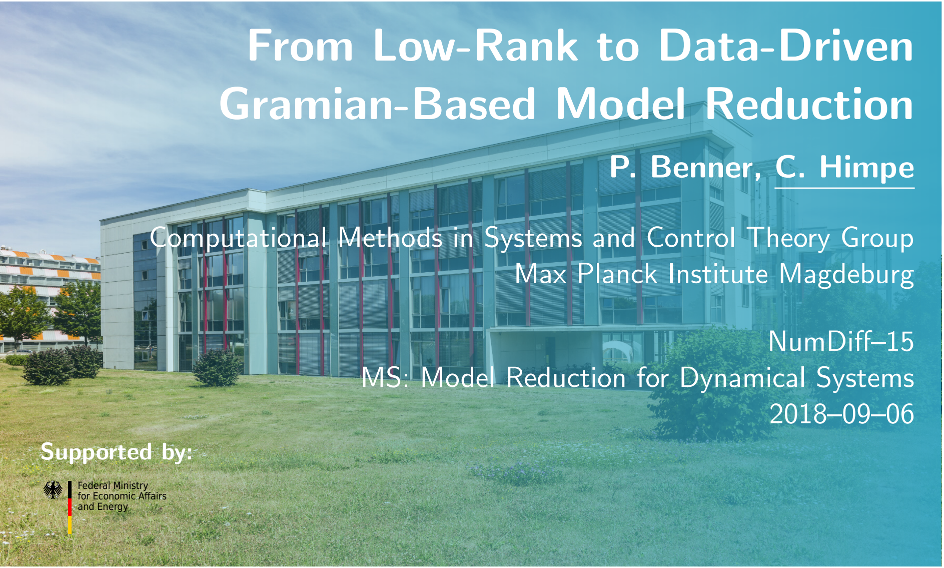
MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

[20 YEARS
1998-2018]



From Low-Rank to Data-Driven Gramian-Based Model Reduction

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NumDiff–15
MS: Model Reduction for Dynamical Systems
2018–09–06

Supported by:

Federal Ministry
for Economic Affairs
and Energy

Superlinear Gramians

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- Those are the best choice for these systems.
- Yet currently, nonlinear Gramians are hard to compute.
- And what about other nonlinear systems?
- Let's approximate nonlinear system Gramians empirically!

Overview

1. From Linear to Nonlinear

2. Empirical Gramians

3. Numerical Experiments

$$\dot{x}(t) = f(t, x(t), u(t), \theta)$$
$$y(t) = g(t, x(t), u(t), \theta)$$

- $u : \mathbb{R} \to \mathbb{R}^M$
- $x : \mathbb{R} \to \mathbb{R}^N$
- $y : \mathbb{R} \to \mathbb{R}^Q$
- $\theta \in \mathbb{R}^P$

Model $\downarrow$ Reduction

$$\dot{x}_r(t) = f_r(t, x_r(t), u(t), \theta)$$
$$\tilde{y}(t) = g_r(t, x_r(t), u(t), \theta)$$

- $x_r : \mathbb{R} \to \mathbb{R}^n$
- $\|y - \tilde{y}\| \ll 1$

Recap: Linear System Gramians

Linear Time-Invariant System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

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Observability Gramian:

$$W_O := \int_0^\infty \mathrm{e}^{A^\intercal t} \, C^\intercal C \, \mathrm{e}^{At} \, \mathrm{d}t$$

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Observability Gramian:

$$W_O := \int_0^\infty \mathrm{e}^{A^\intercal t} C^\intercal C \mathrm{e}^{At} \,\mathrm{d}t$$

Cross Gramian:

$$W_X := \int_0^\infty \mathrm{e}^{At} BC \mathrm{e}^{At} \,\mathrm{d}t$$

Data-Driven Gramians

Impulse response:

$$g(t) = \mathrm{e}^{At} B$$

System Gramians via impulse responses:

$$\begin{aligned} W_C &= \int_0^\infty \mathrm{e}^{At} B B^\intercal \mathrm{e}^{A^\intercal t} \,\mathrm{d}t \\ &= \int_0^\infty (\mathrm{e}^{At} B)(\mathrm{e}^{At} B)^\intercal \,\mathrm{d}t \\ &= \int_0^\infty g(t) g(t)^\intercal \,\mathrm{d}t \end{aligned}$$

Systematic Simulations

Perturbation Sets:

- Directions: $e^z \in \mathbb{R}^Z, e_i^z = \delta_{iz}$
- Scales: $c_k > 0$
- Rotations: $R_\ell \in \mathbb{R}^{M \times Z}, R_\ell^\intercal R_\ell = \mathbb{1}$

Perturbation Types:

- Component Perturbations: $Z = M, \quad R_\ell = \{\mathbb{1}, -\mathbb{1}\}$
- Factorial Perturbations: $Z = 2^M$
- ...

Empirical Gramian Construction

Example: Controllability Gramian

$$W_C = \int_0^\infty \Psi(t)\,\mathrm{d}t$$

$$\Psi(t) = g(t)g(t)^\intercal$$

Empirical Gramian Construction

Example: Controllability Gramian

$$\widehat{W}_C = \int_0^\infty \Psi(t)\,\mathrm{d}t$$

$$\Psi(t) = x(t)x(t)^\intercal$$

Empirical Gramian Construction

Example: Controllability Gramian

$$\widehat{W}_C = \int_0^\infty \Psi(t)\,\mathrm{d}t$$
$$\Psi(t) = (x(t) - \bar{x})(x(t) - \bar{x})^\intercal$$

- $\bar{x} = \lim_{x \to \infty} \frac{1}{T} \int_0^T x(t)\,\mathrm{d}t$

Empirical Gramian Construction

Example: Controllability Gramian

$$\widehat{W}_C = \sum_{m=1}^{M} \int_0^\infty \Psi^m(t)\,\mathrm{d}t$$

$$\Psi^m(t) = (x^m(t) - \bar{x}^m)(x^m(t) - \bar{x}^m)^\intercal$$

- $\bar{x}^m = \lim_{x \to \infty} \frac{1}{T} \int_0^T x^m(t)\,\mathrm{d}t$
- $x^m(t)$ trajectory for $u(t) = e^m \delta(t)$

Empirical Gramian Construction

Example: Controllability Gramian

$$\widehat{W}_C = \frac{1}{K} \sum_{k=1}^{K} \sum_{m=1}^{M} \frac{1}{c_k^2} \int_0^\infty \Psi^{km}(t)\,\mathrm{d}t$$

$$\Psi^{km}(t) = (x^{km}(t) - \bar{x}^{km})(x^{km}(t) - \bar{x}^{km})^\intercal$$

- $\bar{x}^{km} = \lim_{x \to \infty} \frac{1}{T} \int_0^T x^{km}(t)\,\mathrm{d}t$
- $x^{km}(t)$ trajectory for $u(t) = c_k e^m \delta(t)$

Empirical Gramian Construction

Example: Controllability Gramian

$$\widehat{W}_C = \frac{1}{KL} \sum_{k=1}^{K} \sum_{\ell=1}^{L} \sum_{m=1}^{M} \frac{1}{c_k^2} \int_0^\infty \Psi^{k\ell m}(t) \,\mathrm{d}t$$

$$\Psi^{k\ell m}(t) = (x^{k\ell m}(t) - \bar{x}^{k\ell m})(x^{k\ell m}(t) - \bar{x}^{k\ell m})^\intercal$$

- $\bar{x}^{k\ell m} = \lim_{x \to \infty} \frac{1}{T} \int_0^T x^{k\ell m}(t) \,\mathrm{d}t$
- $x^{k\ell m}(t)$ trajectory for $u(t) = c_k R_\ell e^m \delta(t)$

Empirical Controllability Gramian[1]:

$$\widehat{W}_C := \frac{1}{KL} \sum_{k=1}^{K} \sum_{\ell=1}^{L} \sum_{m=1}^{M} \frac{1}{c_k^2} \int_0^\infty \Psi^{k\ell m}(t) \, \mathrm{d}t$$

$$\Psi^{k\ell m}(t) = (x^{k\ell m}(t) - \bar{x}^{k\ell m})(x^{k\ell m}(t) - \bar{x}^{k\ell m})^\intercal$$

- with state trajectories $x^{k\ell m}$ for
- zero initial condition $x(0) = 0$,
- and impulse input $u(t) = c_k R_\ell e^m \delta(t)$.

[1] S. Lall, J.E. Marsden, S. Glavaški. **Empirical Model Reduction of Controlled Nonlinear Systems**. IFAC Proceedings Volumes (Proceedings of the 14th IFAC World Congress) 32(2): 2598–2603, 1999. doi:10.1016/S1474-6670(17)56442-3

About Empirical Gramians

Empirical Gramians are ...

- ... equal to the classic Gramians for linear systems;
- ... not equal to the system Gramians of the linearized system;
- ... time-limited Gramians only if no steady-state is reached;
- ... similar to POD, especially the controllability Gramian;
- ... related to balanced POD, for low-rank empirical Gramians.

Empirical Gramian Overview

Empirical System Gramians[2]:

- Empirical Controllability Gramian
- Empirical Observability Gramian
- Empirical Linear Cross Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

[2] C. H. emgr – **The Empirical Gramian Framework**. Algorithms 11(7): 91, 2018. doi:10.3390/a11070091

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Empirical Cross Gramian

Empirical Cross Gramian[3]:

$$\widehat{W}_X := \frac{1}{KL} \sum_{k=1}^{K} \sum_{\ell=1}^{L} \sum_{m=1}^{M} \frac{1}{c_k d_\ell} \int_0^\infty \Psi^{k\ell m}(t)\,\mathrm{d}t$$

$$\Psi_{ij}^{k\ell m}(t) = (x_i^{km}(t) - \bar{x}_i^{km})(y_m^{\ell j}(t) - \bar{y}_m^{\ell j})$$

- using component perturbations and entailing simplifications.
- Input perturbations for state trajectories x^{km}.
- Initial state perturbations for output trajectories $y^{\ell j}$.

[3] C. H., M. Ohlberger. **Cross-Gramian-Based Combined State and Parameter Reduction for Large-Scale Control Systems**. Mathematical Problems in Engineering 2014: 843869, 2014. doi:10.1155/2014/843869

Low-Rank Empirical Cross Gramian

Partitioned Computation:

$$\widehat{W}_X = \begin{pmatrix} \hat{w}_1 & \dots & \hat{w}_N \end{pmatrix}$$

$$\hat{w}_n = \frac{1}{KL} \sum_{k=1}^{K} \sum_{\ell=1}^{L} \sum_{m=1}^{M} \frac{1}{c_k d_\ell} \int_0^\infty \psi^{k\ell mn}(t)\,\mathrm{d}t$$

$$\psi_i^{k\ell mn}(t) = (x_i^{km}(t) - \bar{x}_i^{km})(y_m^{\ell n}(t) - \bar{y}_m^{\ell n})$$

- Empirical cross Gramian can be computed column-wise.
- Incremental decompositions for a low-rank cross Gramian.
- Hierarchical Approximate Proper Orthogonal Decomposition[4].

[4] C. H. and T. Leibner and S. Rave. **Hierarchical Approximate Proper Orthogonal Decomposition**. SIAM Journal on Scientific Computing: Accepted, 2018.

Sequential Computation:

$$\widetilde{W}_C := \begin{pmatrix} \widetilde{w}_1 & \dots & \widetilde{w}_Q \end{pmatrix}$$
$$\widetilde{w}_q = \frac{1}{c_k(q)^2} \int_0^\infty \psi^{k(q)\ell(q)m(q)}(t)\,\mathrm{d}t$$
$$\psi^{k\ell m}(t) = (x^{k\ell m}(t) - \bar{x}^{k\ell m})(x^{k\ell m}(t) - \bar{x}^{k\ell m})^\intercal$$

- $W_C = W_C^\intercal \geq 0$, $W_O = W_O^\intercal \geq 0$.
- Incremental HAPOD for low-rank SVD of W_C, W_O.
- Balanced POD[5].

[5] C.W. Rowley. **Model Reduction for Fluids, Using Balanced Proper Orthogonal Decomposition**. International Journal of Bifurcation and Chaos 15(3): 997–1013, 2005. doi:10.1142/S0218127405012429

Simulate, POD, Repeat

- Empirical Controllability Gramian HAPOD:
  1. Compute state trajectory x^q.
  2. Compute POD modes of $\begin{pmatrix} x^q & \tilde{x}^{q-1} \end{pmatrix}$.
  3. Scale POD modes with singular values ($\tilde{x}^q$).
- Empirical Observability Gramian HAPOD:
  1. Compute output trajectories y^n.
  2. Compute "right" modes of $\begin{pmatrix} y^n & \tilde{y}^{n-1} \end{pmatrix}$.
  3. Scale "right" modes with singular values ($\tilde{y}^n$).
- Balancing Transformation:
  1. Compute $Z = Y^\intercal X$.
  2. SVD via method of snapshots.

Pipe Model

Isothermal Euler Equations[6]:

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho &= -\frac{1}{S}\frac{\partial}{\partial x}q \\
\frac{\partial}{\partial t}q &= -S\frac{\partial}{\partial x}p - Sg\rho\frac{\partial}{\partial x}h - \frac{\lambda}{2DS}\frac{q|q|}{\rho} \\
p &= R_S T_0 z \rho
\end{aligned}
$$

- Density: $\rho(x, t)$
- Mass-Flux: $q(x, t)$
- Pressure: $p(x, t)$
- Elevation: $h(x)$
- Constants: S, g, D
- Parameters: T_0, R_S
- Friction Factor: $\lambda(q)$
- Compressibility Factor: $z(p, T)$

[6] P. Benner, S. Grundel, C. H., C. Huck, T. Streubel, C. Tischendorf. **Gas Network Benchmark Models**. In: Differential-Algebraic Equation Forum, 2018. doi:10.1007/11221_2018_5

Input-Output System

Decoupled and spatially discretized model:

$$\dot{x}(t) = A(\theta)x(t) + f(x(t), \theta) + Bu(t)$$
$$y(t) = Cx(t)$$

- Networks of pipes $\rightarrow$ Kirchhoff rules[7]
- Always square $\rightarrow$ Cross Gramian
- Friction nonlinearity $\rightarrow$ Empirical Gramians
- Stiff linear part $\rightarrow$ IMEX Solver
- Multi-scale behavior $\rightarrow$ Structured Reduction

[7] T.P. Azevedo-Perdicoúlis and G. Jank. **Modelling Aspects of Describing a Gas Network Through a DAE System**. 3rd IFAC Symposium on System Structure and Control 40(20): 40–45, 2007. doi:10.3182/20071017-3-BR-2923.00007

Test Network and Scenario

Test Network:

- Distribution network
- Tree topology
- Single supply
- Multiple demands
- About 700 DoFs

Test Scenario:

- Parameter setting
- Pressure at supply nodes
- Mass-flow at demand nodes
- $24\,\mathrm{h}$ time horizon
- $1\,\mathrm{min}$ time resolution

Model Reduction Workflow

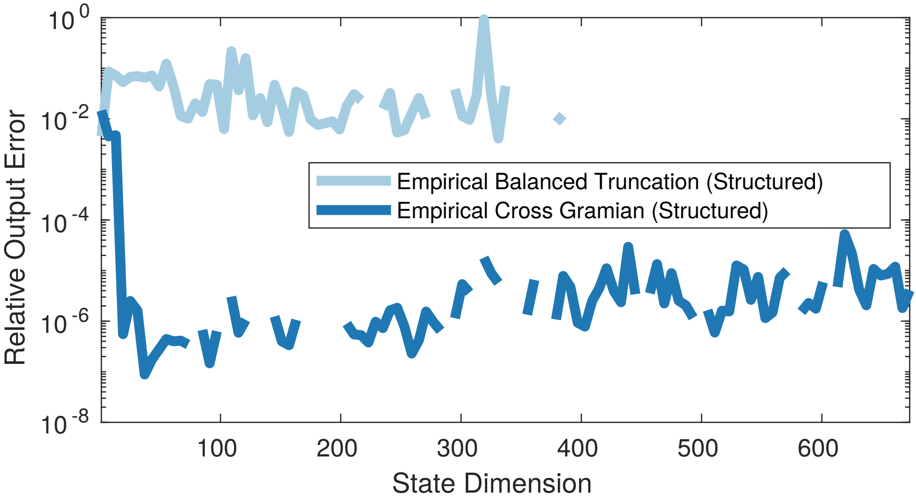
Offline:

1. Design input and parameter samples.
2. Compute empirical Gramians,
3. for short time horizons ($< 1\,\mathrm{h}$).
4. Compute projections from Gramians.

Online:

1. Design scenarios.
2. Simulate scenarios,
3. for long time horizons ($24\,\mathrm{h}$).

L_2 Model Reduction Error



Outlook

- Compare polynomial and empirical Gramians.
- Connect polynomial with empirical Gramians.
- Dominant subspaces[8] instead of balancing.
- System indices for nonlinear systems.
- More automation.

[8] T. Penzl. **Algorithms for model reduction of large dynamical systems**. Linear Algebra and its Applications, 415(2–3): 322–343, 2006. doi:10.1016/j.laa.2006.01.007

Summary

Take-home message:

- Empirical Gramians are
- universal in application and
- simple in computation.

https://himpe.science

Acknowledgment:
Supported by the German Federal Ministry for Economic Affairs and Energy, in the joint project: "**MathEnergy** – Mathematical Key Technologies for Evolving Energy Grids", sub-project: Model Order Reduction (Grant number: 0324019**B**).

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Features:

- Interfaces for: Solver, inner product kernels & low-rank computation
- Open-source OCTAVE and MATLAB toolbox
- Configurable, Vectorized and parallelizable

More info: `https://gramian.de`

[9] C. H. emgr – **The Empirical Gramian Framework**. Algorithms 11(7): 91, 2018. doi:10.3390/a11070091