MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

20 YEARS 1998-2018

Optimal Sensor Placement for Nonlinear Systems

P. Benner, C. Himpe

Computational Methods in Systems and Control Theory Group
Max Planck Institute Magdeburg

TU Braunschweig
2018–06–25

Supported by:

Federal Ministry
for Economic Affairs
and Energy

Optimal Sensor Location

Given an ODE system (i.e. a spatially discretized PDE):

$$\dot{x}(t) = f(x(t))$$

- With $x_i(t)$ describing the temporal evolution
- of a discrete location x_i.
- Where on x does one place sensors,
- so a small number suffices to identify $x \in \mathbb{R}^N$?

Sensor Encoding

Sensor output:

$$y(t) = Cx(t)$$

- Given $Q \geq 1$ sensors.
- (Projected) Permutation matrix $C \in P^{Q \times N}$.
- Each C represents a sensor configuration.
- Example: $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.
- $Q = 1$ sensor at position x_4, $x \in \mathbb{R}^5$.

Linear Observability

Linear System:

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

Observability Operator:

$$\mathcal{O}(x_0)(t) := C\,\mathrm{e}^{At}\,x_0$$

Observability Gramian:

$$W_O := \mathcal{O}^*\mathcal{O} = \int_0^\infty \mathrm{e}^{A^\intercal t}\,C^\intercal C\,\mathrm{e}^{At}\,\mathrm{d}t \in \mathbb{R}^{N \times N}$$
$$\Leftrightarrow A^\intercal W_O + W_O A = -C^\intercal C$$

Observability Measures

- $\mu_0(W_O) := \mathrm{rank}(W_O)$
- $\mu_1(W_O) := \lambda_{\text{min}}(W_O)$
- $\mu_2(W_O) := \lambda_{\text{max}}(W_O)$
- $\mu_3(W_O) := \mathrm{tr}(W_O) = \sum_{i=1}^N \lambda_i(W_O)$
- $\mu_4(W_O) := \mathrm{tr}(W_O^{-1}) = \sum_{i=1}^N \lambda_i^{-1}(W_O)$
- $\mu_5(W_O) := \log\Big(\det(W_O)\Big) = \sum_{i=1}^N \log(\lambda_i(W_O))$
- $\mu_6(W_O) := \mathrm{cond}(W_O) = \frac{\lambda_{\text{max}}(W_O)}{\lambda_{\text{min}}(W_O)}$
- etc.

Optimal Sensor Placement

Note:

$$W_O : P^{Q \times N} \to \mathbb{R}^{N \times N}$$

Optimization Problem:

$$C_0 = \operatorname*{arg\,max}_{C \in P^{Q \times N}} \ \mu_*(W_O(C))$$

- Which $C \in P^{Q \times N}$ maximizes observability?
- Discrete Optimization $\rightarrow$ Integer Programming

From Linear to Nonlinear

Reminder:

$$W_O = \int_0^\infty \mathrm{e}^{A^\intercal t} C^\intercal C \, \mathrm{e}^{At} \, \mathrm{d}t$$
$$= \int_0^\infty (\mathrm{e}^{A^\intercal t} C^\intercal)(\mathrm{e}^{A^\intercal t} C^\intercal)^\intercal \, \mathrm{d}t$$

- $z(t) = \mathrm{e}^{A^\intercal t} C^\intercal$ is impulse responses of adjoint system.
- Impulse responses can be computed empirically.
- Generally, only linear adjoint system accessible.

Empirical Observability Gramian

Nonlinear System:

$$\dot{x}(t) = f(x(t))$$
$$y(t) = Cx(t)$$

Empirical Observability Gramian:

$$\widehat{W}_O = \frac{1}{|S_x|} \sum_{l=1}^{|S_x|} \frac{1}{d_l^2} \int_0^\infty \Psi^l(t)\,\mathrm{d}t \in \mathbb{R}^{N \times N}$$

$$\Psi_{ij}^l(t) = (y^{li}(t) - \bar{y}^{li})^\intercal (y^{lj}(t) - \bar{y}^{lj}) \in \mathbb{R}$$

- y^{li} is the output for the initial condition with l-th perturbation of i-th component
- For linear systems: $\widehat{W}_O = W_O$

Parameter Identifiability

Parametric System:

$$\dot{x}(t) = f(x(t), \theta)$$

Augmented System:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} f(x(t), \theta(t)) \\ 0 \end{pmatrix}$$

Augmented (Empirical) Observability Gramian:

$$\widehat{W}_O^+ = \begin{pmatrix} W_O & W_M \\ W_M^\intercal & W_P \end{pmatrix}$$

Notes

- May become mixed integer optimization.
- May be combined with genetic algorithms.
- May be adapted for unknown number of sensors.
- Acceleration of $\widehat{W}_O$ computation possible.
- Try: `emgr` - EMpirical GRamian Framework (Version 5.4), 2018. `http://gramian.de`

Summary

- Maximize observability for optimal sensor placement.
- Computable approximately for nonlinear systems.
- (Maximize controllability for optimal actuator placement.)

http://himpe.science

Acknowledgment:
Supported by the German Federal Ministry for Economic Affairs and Energy, in the joint project: "**MathEnergy** – Mathematical Key Technologies for Evolving Energy Grids", sub-project: Model Order Reduction (Grant number: 0324019**B**).

References

- P.C. Müller and H.I. Weber. **Analysis and Optimization of Certain Qualities of Controllability and Observability for Linear Dynamical Systems**. Automatica 8: 237–246, 1972. doi:10.1016/0005-1098(72)90044-1
- S. Lall, J.E. Marsden and S. Glavaški. **Empirical model reduction of controlled nonlinear systems**. IFAC Proceedings Volumes 32(2): 2598-2603, 1999. doi:10.1016/S1474-6670(17)56442-3
- A.K. Singh and J. Hahn. **Determining Optimal Sensor Locations for State and Parameter Estimation for Stable Nonlinear Systems**. Industrial & Engineering Chemistry Research 44(15): 5645–5659, 2005. doi:10.1021/ie040212v
- A.K. Singh and J. Hahn. **Sensor Location for Stable Nonlinear Dynamic Systems: Multiple Sensor Case**. Industrial & Engineering Chemistry Research 45(10): 3615–3623, 2006. doi:10.1021/ie0511175
- J. Qi, K. Sun and W. Kang. **Optimal PMU Placement for Power System Dynamic State Estimation by Using Empirical Observability Gramian**. Transaction on Power Systems 30(4): 2041–2054, 2015. doi: 10.1109/TPWRS.2014.2356797
- T.H. Summers, F.L. Cortesi and J. Lygeros. **On Submodularity and Controllability in Complex Dynamical Networks**. IEEE Transactions on Control of Network Systems 3(1): 91–101 , 2016. doi:10.1109/TCNS.2015.2453711
- C. Himpe. **emgr - The Empirical Gramian Framework**. arXiv cs.MS: 1611.00675, 2018.