MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

On Empirical System Gramians

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90th GAMM Annual Meeting
Section 22: Scientific Computing
2019–02–21

Supported by:

Federal Ministry
for Economic Affairs
and Energy

About

Gramian (Matrix):
The matrix of inner products for a set of vectors.

(Input-Output) System:
A dynamical system with time-varying inhomogeneity (input) and a transformation of the state (output).

Empirical:
By means of numerical integration.

Input-Output Systems

Input-Output System:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

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Example: Linear Time-Invariant Systems (LTI)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

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Example: Time-Dependent Partial Differential Equation (PDE)

$$\partial_t z(x,t) + L_x(z(x,t)) = F(x,t)$$
$$y(x,t) = z(x,t)$$

Input-Output Systems

Linear Time-Invariant Systems (LTI)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Reachability

Definition:

A state $\bar{x} \in \mathbb{R}^N$ is reachable from the zero state,
if there exists an input function $\bar{u} \in L_2$
and a time $\bar{T} < \infty$ such that:

$$\bar{x} = \int_0^{\bar{T}} \mathrm{e}^{At} \, B\bar{u}(t) \, \mathrm{d}t.$$

Reachability Operator:

$$\mathcal{R}(u) := \int_0^{\infty} \mathrm{e}^{At} \, Bu(-t) \, \mathrm{d}t$$

Adjoint Reachability Operator:

$$\mathcal{R}^*(z_\infty) := B^* \, \mathrm{e}^{-A^* t} \, z_\infty$$

Observability

Definition:

A state $\bar{x} \in \mathbb{R}^N$ is **un**observable,
if for all time $t > 0$:

$$C\,\mathrm{e}^{At}\,\bar{x} = 0.$$

Observability Operator:

$$\mathcal{O}(x_0) := C\,\mathrm{e}^{At}\,x_0$$

Adjoint Observability Operator:

$$\mathcal{O}^*(v) := \int_\infty^0 \mathrm{e}^{A^*t}\,C^* v(t)\,\mathrm{d}t$$

System Gramians

Reachability Gramian [Kalman'60]:

$$W_R := \mathcal{R} \circ \mathcal{R}^* = \int_0^\infty \mathrm{e}^{At} B B^* \mathrm{e}^{A^* t} \,\mathrm{d}t \in \mathbb{R}^{N \times N}$$

Observability Gramian [Kalman'60]:

$$W_O := \mathcal{O}^* \circ \mathcal{O} = \int_0^\infty \mathrm{e}^{A^* t} C^* C \,\mathrm{e}^{At} \,\mathrm{d}t \in \mathbb{R}^{N \times N}$$

Cross "Gramian" [Fernando & Nicholson'83]:

$$W_X := \mathcal{R} \circ \mathcal{O} = \int_0^\infty \mathrm{e}^{At} B C \,\mathrm{e}^{At} \,\mathrm{d}t \in \mathbb{R}^{N \times N}$$

Hankel Operator: $H := \mathcal{O} \circ \mathcal{R}$

Local Approximation

Local Linearization:

$$\begin{aligned}\widetilde{A}(t) &:= (\partial_x f)(x(t), u(t), x_0) \\ \tilde{b}_i(t) &:= (\partial_{u_i} f)(x(t), u(t), x_0) \\ \tilde{c}_i(t) &:= (\partial_x g_i)(x(t), u(t), x_0)\end{aligned}$$

Local Reachability [Stigter et al.'18]:

$$\widetilde{\mathcal{R}}_i(u) := \int_0^\infty \Phi_{\widetilde{A}}(t)\tilde{b}_i(t)u(t)\,\mathrm{d}t$$

Local Observability [Krener & Ide'09]:

$$\widetilde{\mathcal{O}}_i(X_0) := \tilde{c}_i(t)\Phi_{\widetilde{A}}(t)X_0, \quad X_0 = \begin{bmatrix} x_0^1 & \dots & x_0^N \end{bmatrix}$$

Empirical System Gramians

Empirical Reachability Gramian [Lall et al.'99]:

$$\widetilde{W}_R := \sum_i \widetilde{\mathcal{R}}_i \circ \widetilde{\mathcal{R}}_i^* \in \mathbb{R}^{N \times N}$$

Empirical Observability Gramian [Lall et al.'99]:

$$\widetilde{W}_O := \sum_i \widetilde{\mathcal{O}}_i^* \circ \widetilde{\mathcal{O}}_i \in \mathbb{R}^{N \times N}$$

Empirical Cross Gramian [Streif et al.'06]:

$$\widetilde{W}_X := \sum_i \widetilde{\mathcal{R}}_i \circ \widetilde{\mathcal{O}}_i \in \mathbb{R}^{N \times N}$$

Applications

- Model Reduction
- Combined State and Parameter Reduction
- Decentralized Control
- Sensitivity Analysis
- Parameter or Structural Identifiability
- Optimal Actuator and Sensor Placement
- Nonlinearity Quantification
- System Characterization via Invariants and Indices
- etc.

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Gramian-Based Model Reduction

- Proper Orthogonal Decomposition
- Balanced Proper Orthogonal Decomposition
- Balanced Truncation
- Frequency-/Time-Limited Balanced Truncation
- Frequency-/Time-Weighted Balanced Truncation
- Balanced Gains
- Approximate Balancing
- Dominant Subspaces Projection Model Reduction
- etc.

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- **Proper Orthogonal Decomposition**
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- **Dominant Subspaces Projection Model Reduction**
- etc.

Full Order Model:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

Petrov-Galerkin Projections $(n \ll N)$**:**

$$U : \mathbb{R}^n \to \mathbb{R}^N$$
$$V : \mathbb{R}^N \to \mathbb{R}^n$$

Reduced Order Model:

$$\dot{x}_r(t) = V f(U x_r(t), u(t))$$
$$\tilde{y}(t) = \quad g(U x_r(t), u(t))$$

Example: Linear Transport

1D Linear Transport Equation:

$$\partial_t z = -a\,\partial_x z, \quad z(0,t) = u(t), \quad y(t) = z(1,t)$$

- Hyperbolic PDE
- Velocity $a > 0$
- Spatial: Upwind (# DoFs: $\sim 10^3$)
- Temporal: First Order Runge-Kutta
- Training: Constant step function
- Test: Smooth function

Compared MOR Methods:

- Proper Orthogonal Decomposition: W_R vs $\widetilde{W}_R$
- Balanced Truncation: [Moore'81] vs [Lall et al.'99]
- Dominant Subspaces: [Penzl'06] vs [Benner & H.'18]

Results: Linear Transport

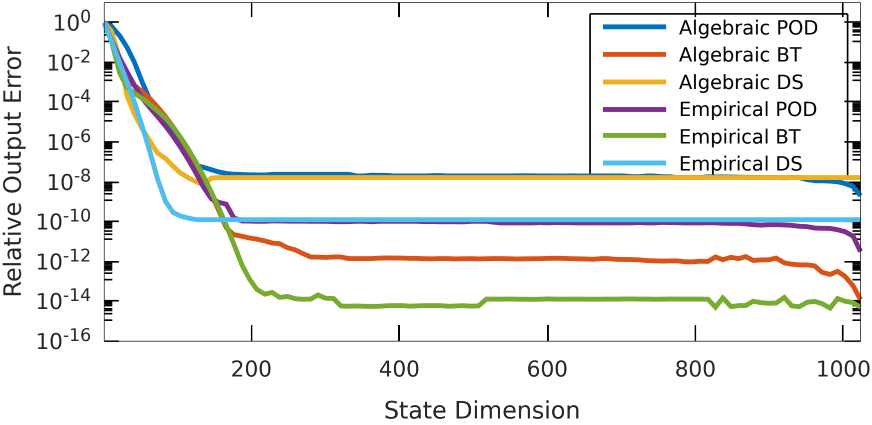

Figure: Relative L_2 Model reduction error for increasing reduced order model state-space dimension of the one-dimensional linear transport equation.

Example: Gas Pipeline

1D Isothermal Euler Equations [Benner et al.'18]:

$$\partial_t p = -\frac{\gamma_0 z_0^2}{S} \partial_x q, \qquad p(0,t) = u_s(t)$$

$$\partial_t q = -S \partial_x p - \frac{Sg}{\gamma_0 z_0} p \partial_x h - \frac{\lambda_0 \gamma_0 z_0}{2DS} \frac{q|q|}{p}, \qquad q(L,t) = u_d(t)$$

$$y(t) = \begin{pmatrix} p(L,t) \\ q(0,t) \end{pmatrix}$$

- Hyperbolic PDE
- Nonlinear (Friction)
- Coupled (Pressure, Mass-Flux)
- Spatial: Midpoint (# DoFs: $\sim 10^3$)
- Temporal: First Order Implicit-Explicit Runge-Kutta
- Training: Delta impulse
- Test: Varying step function

Results: Gas Pipeline

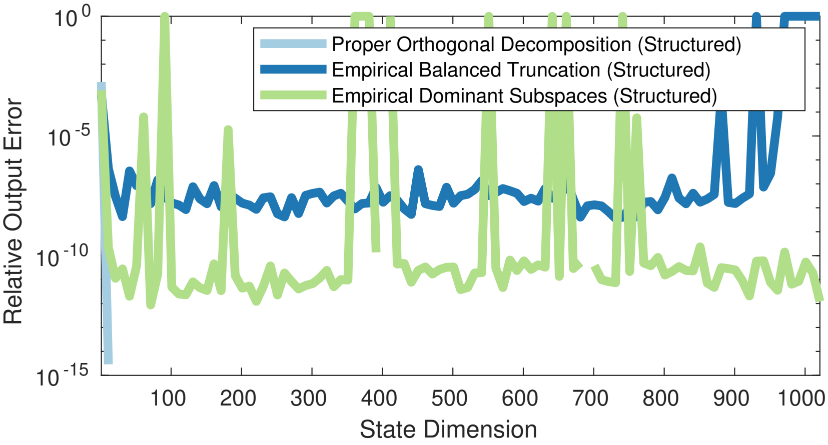

Figure: Relative L_2 Model reduction error for increasing reduced order model state-space dimension of the one-dimensional isothermal Euler equations.

Empirical Gramians

- Empirical Reachability Gramian
- Empirical Observability Gramian
- Empirical Linear Cross Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

Features:

- Open-source OCTAVE and MATLAB toolbox, PYTHON support
- Interfaces for: Solver, inner product kernels & low-rank computation
- Configurable, vectorized and parallelizable

More info: https://gramian.de

C.H. emgr – **The Empirical Gramian Framework**. Algorithms 11(7): 91, 2018. doi:10.3390/a11070091

Summary

- System Gramians for linear input-output systems.
- Empirical system Gramians for nonlinear input-output systems.
- Empirical Gramians can be advantageous even for linear systems.
- ▶ Specialized empirical Gramians for hyperbolic input-output sytems?
- ▶ What training variants work best for what systems?

https://himpe.science

Acknowledgment:
Supported by the German Federal Ministry for Economic Affairs and Energy, in the joint project: "**MathEnergy** – Mathematical Key Technologies for Evolving Energy Grids", sub-project: Model Order Reduction (Grant number: 0324019**B**).