MAX PLANCK INSTITUTE
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TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

DMD, System Identification, and MORe

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Outline

- DMD Basics
- DMD Variants
- DMD-Based System Identification

- DMD-Based Model Reduction

Things to know:

- Foundations of DMD are old [Koopman'31].
- Recently "rediscovered" [Mezic'05].
- Practical computation [Schmid,Sesterhenn'10].
- DMD is a system identification method,
- as well as a model reduction technique,
- as well as a hyper reduction method [Alla,Kutz'17].
- Meanwhile a myriad of variants sprouted.
- This talk is about ioDMD [Benner,H.,Mitchell'18],
- and a guide through DMD methodology.

(Plain) DMD

Autonomous Dynamical System:

$$\dot{x}(t) = f(x(t))$$

Discrete-Time Linear Approximation ($x_k = x(t_k)$):

$$x_{k+1} = Ax_k$$

Given Trajectory Data (i.e. by Runge-Kutta):

$$X = \begin{bmatrix} x_0 & x_1 & \dots & x_K \end{bmatrix}$$

"Plain" DMD [Schmid'10]:

$$\begin{aligned} X_0 &:= \begin{bmatrix} x_0 & x_1 & \dots & x_{K-1} \end{bmatrix} \\ X_1 &:= \begin{bmatrix} x_1 & x_2 & \dots & x_K \end{bmatrix} \\ \rightarrow X_1 &= AX_0 \\ \rightarrow A &= X_1 X_0^+ \end{aligned}$$

Note:

- This assumes linear dynamics!
- Still applicable to “nonlinear data”.
- Outcome depends heavily on quality of data.
- DMD is not energy-based, like POD,
- but instead based on oscillations.

Computation [Tu et al’14], [Alla,Kutz’17]:

1. $X_0 \overset{\text{tSVD}}{=} U_0 \Sigma_0 V_0^*$
2. $\hat{A} := U_0^* X_1 V_0 \Sigma_0^{-1}$
3. $\hat{A} W = \Lambda W$
4. $\Psi = X_1 V_0 \Sigma_0^{-1} W$
5. $x(t) \approx \Psi \operatorname{diag}(\exp(\Lambda t)) \Psi^+ x_0$

- Hyper reduction: $f(x(t)) \approx \Psi \operatorname{diag}(\exp(\Lambda t)) \Psi^+ f(x_0)$

Regularized DMD

DMD is a Least-Squares Fit:

$$\hat{A} = \operatorname*{arg\,min}_{A \in \mathbb{R}^{N \times N}} \|X_1 - AX_0\|_F^2$$

- The optimization problem may be ill-posed.
- Regularization can help.
- Regularization via TSVD [Hansen'87].
- which is equivalent to compressed DMD:

$$\hat{A} = U_0^* A U_0 = U_0^* X_1 V_0 \Sigma_0^{-1} U_0^* U_0 = U_0^* X_1 V_0 \Sigma_0^{-1}$$

Koopman Operator

Discrete-Time Dynamical System:

$$x_{k+1} = f(x_k), \;\; x_k \in \mathcal{M}, \;\; f : \mathcal{M} \to \mathcal{M}$$

Observable:

$$y_k = g(x_k), \;\; g : \mathcal{M} \to \mathbb{R}$$

Koopman Operator:

$$\mathcal{K}(g(x_k)) := g(f(x_k)) = g(x_{k+1})$$

- The Koopman operator is linear,
- but infinite-dimensional,
- even though f maybe nonlinear.

Justifying DMD

Koopman Modes:

$$\mathcal{K}\varphi_i(x) = \lambda_i \varphi_i(x)$$

Koopman Decomposition:

$$g(x) = \sum_{j=1}^{\infty} \varphi_j(x) \langle \varphi_j, g \rangle$$

- Vector-valued observables by stacking.
- Koopman modes describe oscillations,
- Koopman eigenvalues their growth / decay.
- DMD modes are Koopman modes,
- if data is linearly consistent [Tu et al'14].

Exact DMD

Definition [Tu et al'14]:
Given two data sets $X = \begin{bmatrix} x_0 & \dots x_K \end{bmatrix}$ and $Y = \begin{bmatrix} y_0 & \dots y_K \end{bmatrix}$, the *exact DMD* operator is given by:

$$A := YX^+.$$

- Generalizes DMD.
- For $y_k = x_{k+1}$, the exact DMD corresponds to the plain DMD.
- Koopman theory extend respectively.

DMD with Control (DMDc)

Controlled Dynamical System:

$$\dot{x}(t) = f(x(t), u(t))$$

Discrete-Time Linear Approximation ($u_k = u(t_k)$, $x_k = x(t_k)$):

$$x_{k+1} = Ax_k + Bu_k$$

Additionally, Given Input Data:

$$U = \begin{bmatrix} u_0 & u_1 & \dots & u_K \end{bmatrix}$$

DMDc [Proctor,Brunton,Kutz'16]:

$$\begin{aligned} U_0 &:= \begin{bmatrix} u_0 & u_1 & \dots & u_{K-1} \end{bmatrix} \\ \rightarrow X_1 &= AX_0 + BU_0 \\ \rightarrow \begin{bmatrix} A & B \end{bmatrix} &= X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^+ \end{aligned}$$

Koopman with Inputs and Control (KIC)

Koopman Operator with Input u [Proctor,Brunton,Kutz'18]:

$$\mathcal{K}(g(x_k, u_k)) = g(f(x_k, u_k), u_{k+1})$$

Discrete-Time Linear Approximation $(u_k = u(t_k),\ x_k = x(t_k))$:

$$x_{k+1} = Ax_k + Bu_k$$

Linear Dynamics Assumption:

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} G_{xx} & G_{xu} \\ G_{ux} & G_{uu} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}$$

$\rightarrow$ DMDc is special case of KIC.

Input-Output DMD (ioDMD)

Input-Output System:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

Discrete-Time Linear Approximation ($u_k = u(t_k)$, $x_k = x(t_k)$, $y_k = y(t_k)$):

$$x_{k+1} = Ax_k + Bu_k,$$
$$y_k = Cx_k + Du_k$$

Additionally Given Output Data:

$$Y = \begin{bmatrix} y_0 & y_1 & \dots & y_K \end{bmatrix}$$

ioDMD [Annoni,Gebraad,Seiler'16], [Annoni,Seiler'17], [Benner,H.,Mitchell'18]:

$$Y_0 := \begin{bmatrix} y_0 & y_1 & \dots & y_{K-1} \end{bmatrix}$$

$$\rightarrow \begin{cases} X_1 = AX_0 + BU_0 \\ Y_0 = CX_0 + DU_0 \end{cases}$$

$$\rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_0 \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^+$$

N4SID

Numerical Sub-Space State System Identification:

- ioDMD is **Direct** N4SID [Benner,H.,Mitchell'18]
- N4SID invented in [Van Overschee,De Moor'92]
- Direct N4SID [Viberg'95]
- **Reduced Direct N4SID** [Lee'00]
- Overview in [Katayama'05]
- Linear Predictor [Korda,Mezic'18]

Reduced ioDMD

Large-Scale Problem:

$$\begin{bmatrix} X_1 \\ Y_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$$

State Trajectory Data Compression (i.e., via POD):

$$\begin{aligned} Q \; &= \mathrm{POD}(X, \varepsilon), \;\; Q^* Q = I \\ X_r &:= Q^* X \;\to\; X \approx Q X_r \end{aligned}$$

Reduced ioDMD [Benner,H.,Mitchell'18]:

$$\begin{bmatrix} X_{r,1} \\ Y_0 \end{bmatrix} \begin{bmatrix} X_{r,0} \\ U_0 \end{bmatrix}^+ = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}$$

Stabilized DMD

Stabilization via Optimization [Amsallem,Farhat'12], [Benner,H.,Mitchell'18]:

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \operatorname*{arg\,min}_{A,B,C,D} \left\| \begin{bmatrix} X_{r,1} \\ Y_0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_{r,0} \\ U_0 \end{bmatrix} \right\|_F^2 \quad : \quad \lambda(A) < (1 - \tau)$$

- Post-processing step,
- feasible due to reduced order;
- non-convex, but almost everywhere smooth problem,
- solve via BFGS-SQP [Curtis,Mitchell,Overton'17];
- software: GRANSO (`http://www.timmitchell.com/software/GRANSO`)

- Note that a D feedthrough matrix appears.
- This can be exploited for DC gain matching.

Recap

DMD Variants:

- Plain DMD (= Exact DMD)
- Compressed DMD (= Regularized DMD)
- DMD with Control (= KIC)
- Input-Output DMD (= Direct N4SID)
- Stabilized DMD

Linear Advection

One-Dimensional Transport Equation:

$$\begin{aligned} \frac{\partial}{\partial t} z(x,t) &= a \frac{\partial}{\partial x} z(x,t) \\ z(0,t) &= u(t) \\ z(x,0) &= 0 \\ y(t) &= z(1,t) \end{aligned}$$

- Linear SISO system,
- but purely hyperbolic,
- with positive velocity: $a > 0$.

(Persistent) Excitation

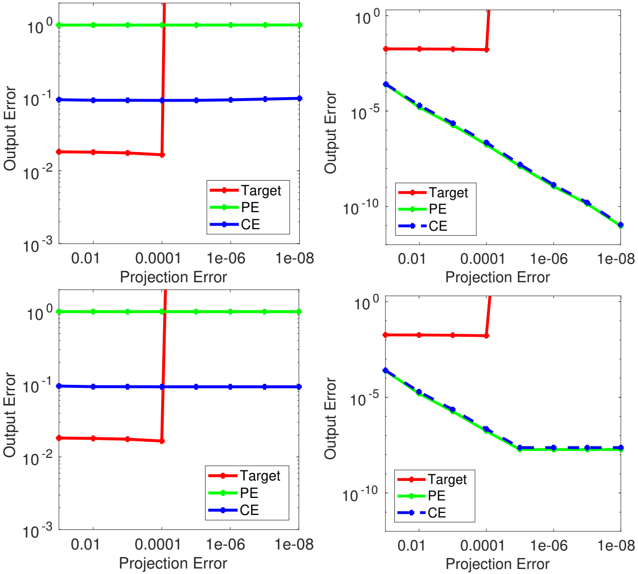
Critical Issue:

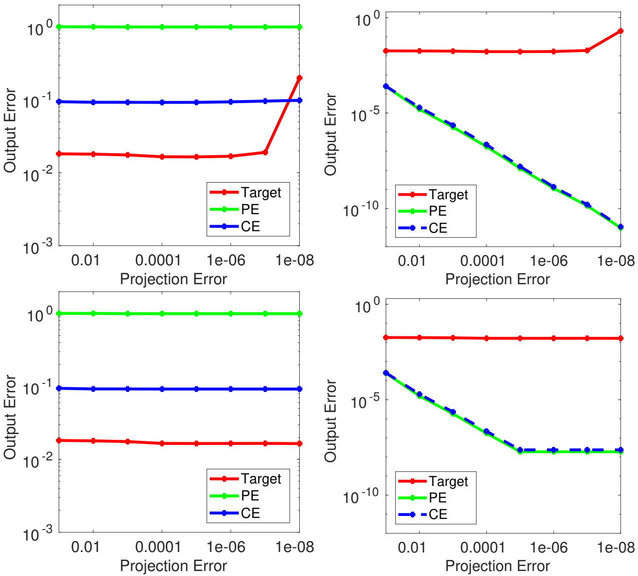
- For data-driven system identification,
- how to excite the system,
- in a scenario-free, generic manner?

Input Choices:

- Impulse (Not a good choice)
- Chirps
- Step
- White Noise
- Pseudo-Random Binary
- Cross Excitation ("Closed Loop") [Benner,H.,Mitchell'18]

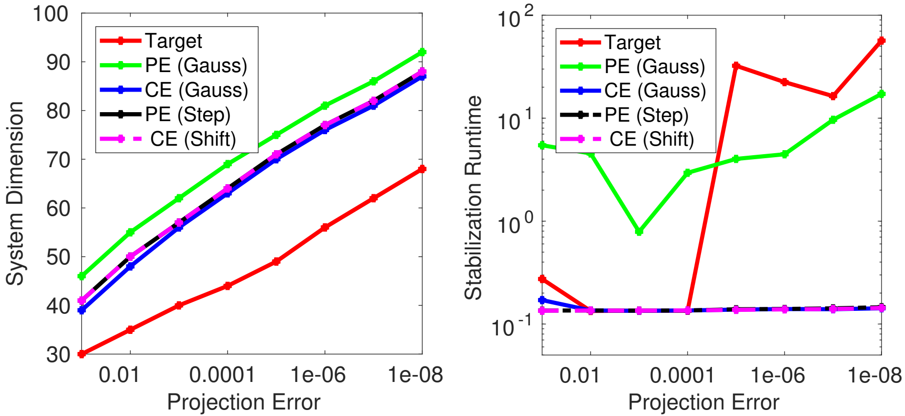
Numerical Results (Noise vs Step)





Output Error
Projection Error
0.01
0.0001
1e-06
1e-08
Target
PE
CE

Numerical Results (Excitation)

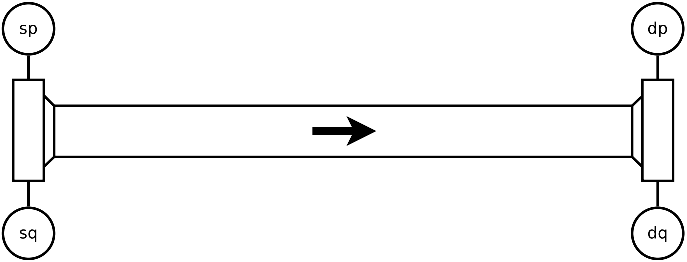


The Gas Network Situation:

- Volatile renewable energies.
- Fluctuating supply and demand.
- Fast response of gas-fired plants.
- Day-ahead forecasts.
- Many simulations before dispatch.

→ MathEnergy model reduction sub-project.

Gas Flow in a Pipe:



- Modeled by Euler equations,
- coupling mass-flow and pressure.
- For example:
  - Boundary values: s_p, d_q,
  - Quantites of Interest: s_q, d_p.

Gas Pipeline Model

(Isothermal) Euler Equations [Benner,Grundel,H.,Huck,Streubel,Tischendorf'18]:

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho &= -\frac{1}{S}\frac{\partial}{\partial x}q \\
\frac{\partial}{\partial t}q &= -S\frac{\partial}{\partial x}p - Sg\rho\frac{\partial}{\partial x}h - \frac{\lambda}{2DS}\frac{q|q|}{\rho} \\
p &= R_S T_0 z \rho
\end{aligned}
$$

- Density: $\rho(x,t)$
- Mass-Flux: $q(x,t)$
- Pressure: $p(x,t)$
- Elevation: $h(x)$
- Constants: S, g, D
- Parameters: T_0, R_S
- Friction Factor: $\lambda(q)$
- Compressibility Factor: $z(p,T)$

Input-Output System

Spatial Midpoint Discretization [Grundel,Jansen,Hornung,Clees,Tischendorf,Benner'14]:

$$\begin{pmatrix} E_p & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & A_p \\ A_q & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 & B_p \\ B_q & 0 \end{pmatrix} \begin{pmatrix} s_p \\ d_q \end{pmatrix} + \begin{pmatrix} 0 \\ f_q(p, q, s_p, d_q) \end{pmatrix}$$

$$\begin{pmatrix} s_q \\ d_p \end{pmatrix} = \begin{pmatrix} 0 & C_q \\ C_p & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

- Square system (each boundary node induces an input and output)
- two-dimensional (but equilibrated scales)
- stiff, non-normal system matrix (due to hyperbolicity)
- non-singular mass matrix (needs index reduction for non-tree networks)
- system with repeated scalar nonlinearities (friction term)

DMD-Galerkin

DMD-Galerkin:

- The DMD modes span a useful subspace.
- Why not use it as a (reduced) basis?
- Proposed in [Alla,Kutz'17].

Orthogonalize DMD Modes:

$$\Psi \overset{\text{SVD}}{=} U \Sigma V^*$$

Use U as Galerkin Projection:

$$\begin{aligned} x_r(t) &:= U^* x(t) \\ \rightarrow x_r(0) &:= U^* x_0 \\ \rightarrow \dot{x}_r(t) &= U^* f(U x_r(t)) \end{aligned}$$

- On top, one could use DMD for hyper-reduction (DMD-DMD)

Gas Pipeline

Pipeline Specification:

- 50km length
- 1m diameter

Workflow:

- Training: 1hr virtual time
- Test: 24hr virtual time

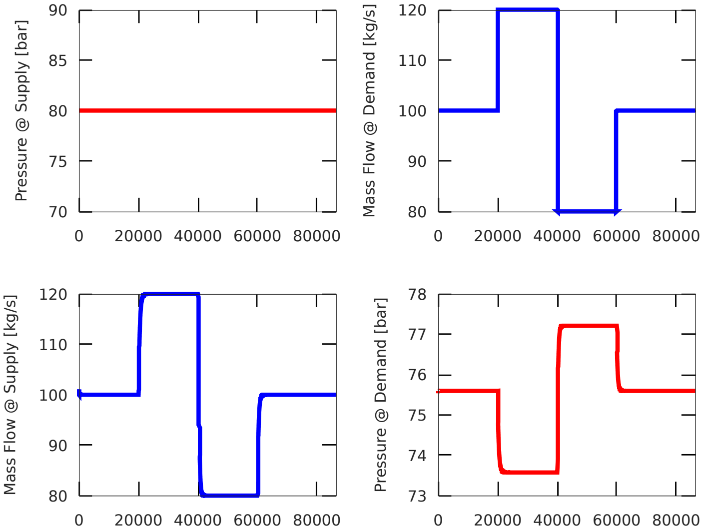
Model Reduction:

- POD
- DMD-Galerkin
- Structured Projection
- Step Excitation

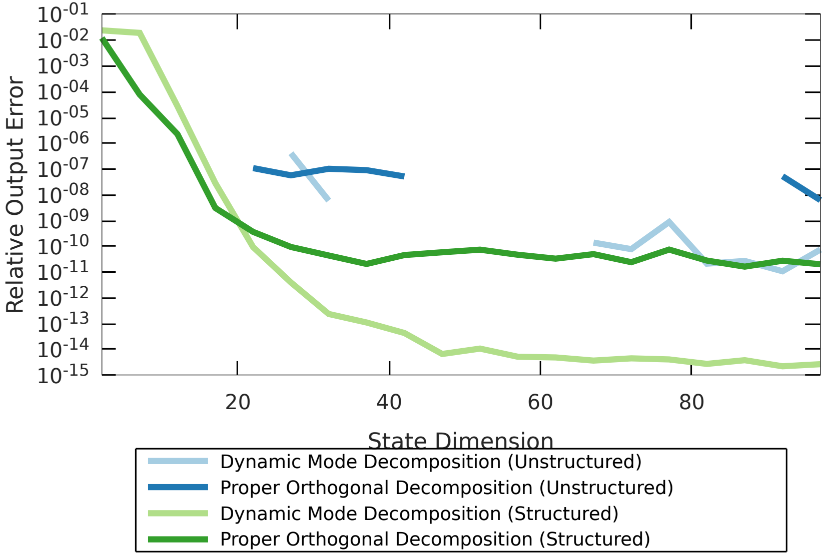
Parametric Model Reduction:

- Temperature: $15°$–$25°$C
- Spec. Gas Constant: 1510–$1550 \frac{J}{mol\,K}$
- Training Samples: 5 (Sparse Grid)
- Test Samples: 10 (Uniformly Random)

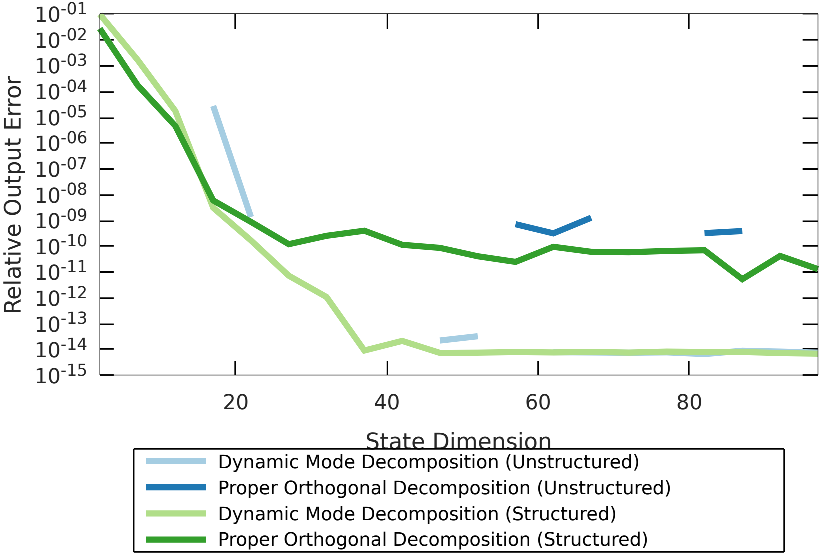
Scenario Simulation



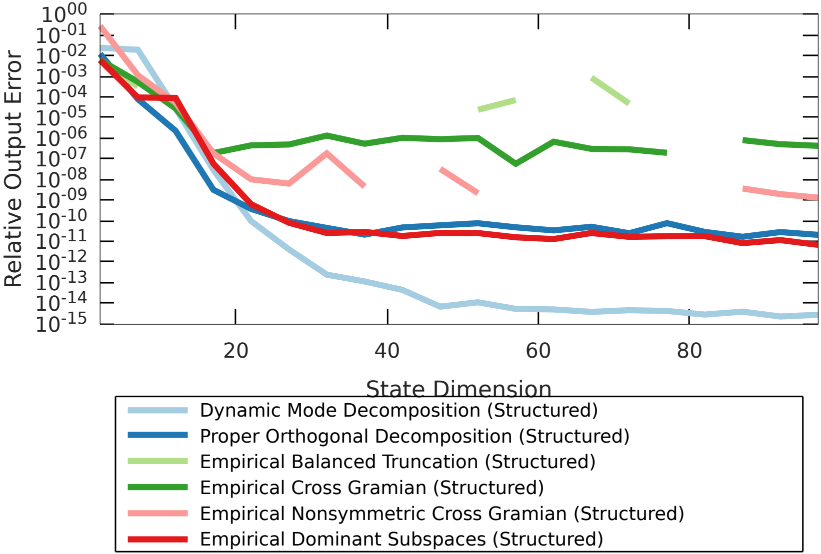
Model Reduction Results



Parametric Model Reduction Results



More Model Reduction Results



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Summary

- DMD is data-driven (reduced order) modelling.
- Stabilized ioDMD for i/o system identification.
- DMD-Galerkin for parametric model reduction.

P. Benner, C. Himpe, T. Mitchell. **On Reduced Input-Output Dynamic Mode Decomposition**. Advances in Computational Mathematics, 44(6): 1751–1768, 2018.
https://doi.org/10.1007/s10444-018-9592-x

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