MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Empirical Dominant Subspaces

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Computational Methods in Systems and Control Theory Group
Max Planck Institute Magdeburg

4th Workshop on Model Reduction of Complex Dynamical Systems (ModRed)
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and Energy

Outline

1. We revisit the DSPMR method.
2. We adapt the empirical cross Gramian.
3. We discuss efficient computation.
4. We derive an a-priori error indicator.
5. We assess numerical examples.

Linear Setting

Generalized Linear Input-Output System:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

- Input: $u : \mathbb{R} \to \mathbb{R}^M$,
- State: $x : \mathbb{R} \to \mathbb{R}^N$,
- Output: $y : \mathbb{R} \to \mathbb{R}^Q$,

- Pencil (A, E) asymptotically stable $(\det(E) \neq 0)$.

Model Reduction

Reduced Order Model:

$$E_r\,\dot{x}_r(t) = A_r\,x_r(t) + B_r\,u(t),$$
$$\tilde{y}(t) = C_r\,x_r(t),$$

Projection-Based Model Reduction:

$$(V_1\;E\;U_1)\,\dot{x}_r(t) = (V_1\;A\;U_1)\,x_r(t) + (V_1\;B)\,u(t),$$
$$\tilde{y}(t) = \quad (C\;U_1)\,x_r(t),$$

(Petrov-)Galerkin Projection:

$$U_1 : \mathbb{R}^N \to \mathbb{R}^n,$$
$$V_1 : \mathbb{R}^n \to \mathbb{R}^N,$$
$$V_1\,U_1 = I_n$$

Generalized Controllability Gramian:

$$W_C := \int_0^\infty \left(\mathrm{e}^{E^{-1}At} E^{-1} B \right) \left(\mathrm{e}^{E^{-1}At} E^{-1} B \right)^* \mathrm{d}t$$

Generalized Observability Gramian:

$$W_O := \int_0^\infty \left(C\, \mathrm{e}^{E^{-1}At} E^{-1} \right)^* \left(C\, \mathrm{e}^{E^{-1}At} E^{-1} \right) \mathrm{d}t$$

Generalized Cross "Gramian":

$$W_X := \int_0^\infty \left(\mathrm{e}^{E^{-1}At} E^{-1} B \right) \left(C\, \mathrm{e}^{E^{-1}At} E^{-1} \right) \mathrm{d}t$$

R.E. Kalman. **Contributions to the Theory of Optimal Control**. Boletin Sociedad Matematica Mexicana 5: 102–119, 1960.
K.V. Fernando, H. Nicholson. **On the Structure of Balanced and Other Principal Representations of SISO Systems**. IEEE Transactions on Automatic Control, 28(2): 228–231, 1983.

Dominant Subspace Projection Model Reduction (DSPMR):

$$
\begin{aligned}
W_C &\overset{\text{Chol}}{=} L_C L_C^\intercal \\
W_O &\overset{\text{Chol}}{=} L_O L_O^\intercal \\
\begin{bmatrix} L_C & L_O \end{bmatrix} &\overset{\text{SVD}}{=} U D V^\intercal \\
U_1 &:= U \\
V_1 &:= U^\intercal
\end{aligned}
$$

Refined DSPMR:

$$
\begin{aligned}
\begin{bmatrix} (\|L_C\|_F^{-1} L_C) & (\|L_O\|_F^{-1} L_O) \end{bmatrix} &\overset{\text{SVD}}{=} U D V^\intercal \\
U_1 &:= U \\
V_1 &:= U^\intercal
\end{aligned}
$$

T. Penzl. **Algorithms for Model Reduction of Large-Scale Systems**. Linear Algebra and its Application, 2–3: 322–343, 2006. (Reprint of 1999 Technical Report)

1. **Gramian Computation and Decomposition:**

$$W_X \overset{\text{tSVD}}{=} U_X D_X V_X^\intercal$$

2. **Subspace Concatenation and Orthogonalization:**

$$\big[(U_X D_X) \;\; (V_X D_X)\big] \overset{\text{rrSVD}}{=} U D V^\intercal$$

3. **Galerkin Projection:**

$$U_1 := U$$
$$V_1 := U^\intercal$$

C.H., P. Benner. **Cross-Gramian-Based Dominant Subspaces**. arXiv, math.OC: 1809.08066, 2019.

Dominant Subspaces

Variants of DSPMR	Gramian Decomposition	Subspace Orthogonalization
[Penzl'06]	Cholesky	rr-SVD
[Li & White'99]	SVD	rr-QR
[Benner & H.'19]	tSVD	rr-SVD

J.-R. Li, J. White. **Efficient Model Reduction of Interconnect via Approximate System Gramians**. 1999 IEEE/ACM International Conference on Computer-Aided Design: 380–383, 1999.

Fused Computation

Wait a second ...

1. I compute an SVD,
2. of the direct sum of (scaled) singular vectors,
3. I care only about the resulting left singular vectors.

C.H., T. Leibner, S. Rave. **Hierarchical Approximate Proper Orthogonal Decomposition**. SIAM Journal on Scientific Computing 40(5): A3267–A3297, 2018.

C.H., T. Leibner, S. Rave, J. Saak. **Fast Low-Rank Empirical Cross Gramians**. Proceedings in Applied Mathematics and Mechanics 17: 841–842, 2017.

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$\rightarrow$ POD of PODs: **HAPOD**!

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$\rightarrow$ POD of PODs: **HAPOD**!

- I do not want to compute a full cross Gramian.
- The empirical cross Gramian can be computed column-wise.
- The HAPOD works column-wise partitioned data.

C.H., T. Leibner, S. Rave. **Hierarchical Approximate Proper Orthogonal Decomposition**. SIAM Journal on Scientific Computing 40(5): A3267–A3297, 2018.

C.H., T. Leibner, S. Rave, J. Saak. **Fast Low-Rank Empirical Cross Gramians**. Proceedings in Applied Mathematics and Mechanics 17: 841–842, 2017.

Empirical Cross Gramian

Empirical Cross Gramian Matrix:

$$\widehat{W}_X := \frac{1}{M} \sum_{m=1}^{M} \int_0^\infty \Psi^m(t) \,\mathrm{d}t \in \mathbb{R}^{N \times N}$$

$$\Psi_{ij}^m(t) = (x_i^m(t) - \bar{x}_i^m)(y_m^j(t) - \bar{y}_m^j) \in \mathbb{R}$$

Column-Wise Computation:

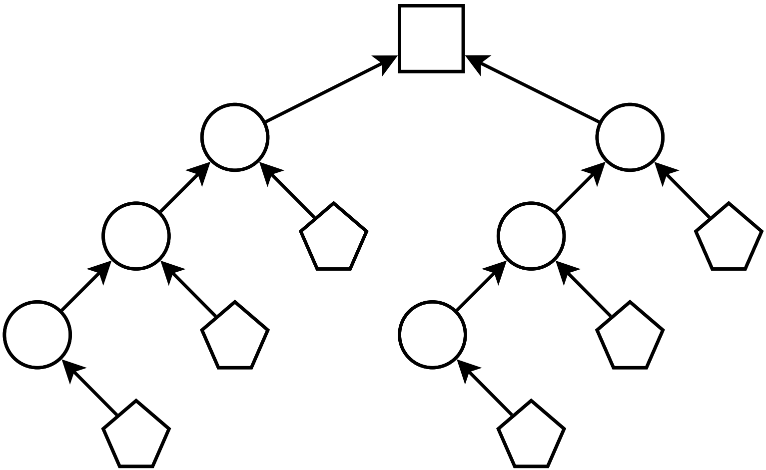
$$W_X = \begin{pmatrix} w_{X,1} & w_{X,2} & \dots & w_{X,N} \end{pmatrix} \in \mathbb{R}^{N \times N}$$

$$w_{X,j} = \frac{1}{M} \sum_{m=1}^{M} \int_0^\infty \psi^{jm}(t) \,\mathrm{d}t \in \mathbb{R}^N$$

$$\psi_i^{jm}(t) = (x_i^m(t) - \bar{x}_i^m)(y_m^j(t) - \bar{y}_m^j) \in \mathbb{R}$$

This can be done only with the empirical **cross** Gramian!

Distributed-Incremental HAPOD



Error Indicator

A-Priori Error Indicator:

$$\|y - \tilde{y}\|_{L_2} \lessapprox \sqrt{\|B\|_2 \|C\|_2 \sqrt{\sum_{k=n+1}^{N} \sigma_k^2(W_X)}}$$

Derivation:

1. H_2 error of (SISO) error system: $\|y - \tilde{y}\|_{H_2}^2 = \mathrm{tr}(\int_0^\infty (C_e \, \mathrm{e}^{A_e} \, B_e)^2)$
2. Matrix exponential Approximation: $\mathrm{e}^{AUU^\intercal}, \mathrm{e}^{UU^\intercal A} \approx \mathrm{e}^A$
3. von-Neumann trace bound: $\mathrm{tr}(AB) \leq \sum_k \sigma_k(A) \sigma_k(B)$

Features:

- Projection error: $\frac{1}{N} \sum_k \|(I - UU^\intercal) W_{X,*k}\|^2 = \sum_{k=n+1}^{N} \sigma_k^2(W_X) \leq \varepsilon^2$
- Only a-priori quantites: B, C, ε
- Time- and frequency-domain relevance

Empirical-Cross-Gramian-Based
Dominant Subspace Projection Model Reduction

1. Compute low-rank empirical cross Gramian W_X.
2. Predict reduction error via projection error of W_X.
3. Compute HAPOD of left and right subspaces.

Convective Thermal Flow Benchmark:

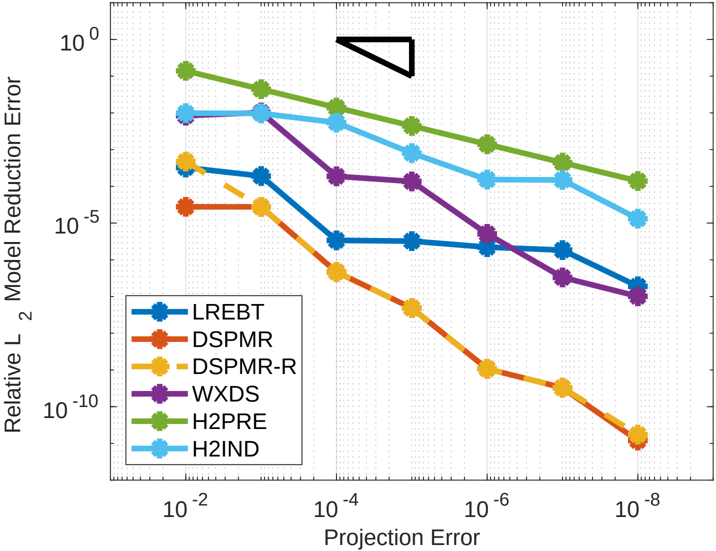
$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T - v \nabla T + \dot{q}$$
$$y = \mathcal{C} T$$

- Temperature T
- **Fluid speed** $v = \{0, \frac{1}{2}\}$
- Thermal conductivity κ (fixed)
- Heat generation rate $\dot{q}$ (fixed)
- Spatial finite element discretization **(SIMO, N=9669)**:

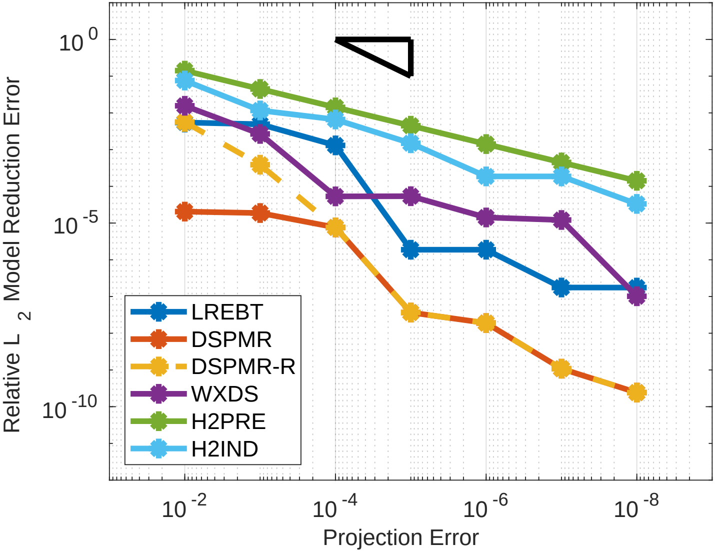
$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

The MORwiki Community. **Convection**. MORwiki – Model Order Reduction Wiki.
`https://modelreduction.org/index.php/Convection`

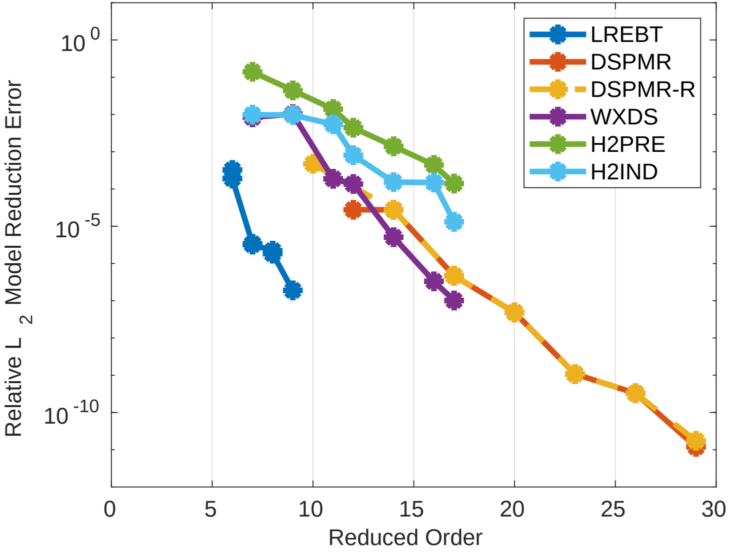
Projection Error vs Output Error $(v = 0)$



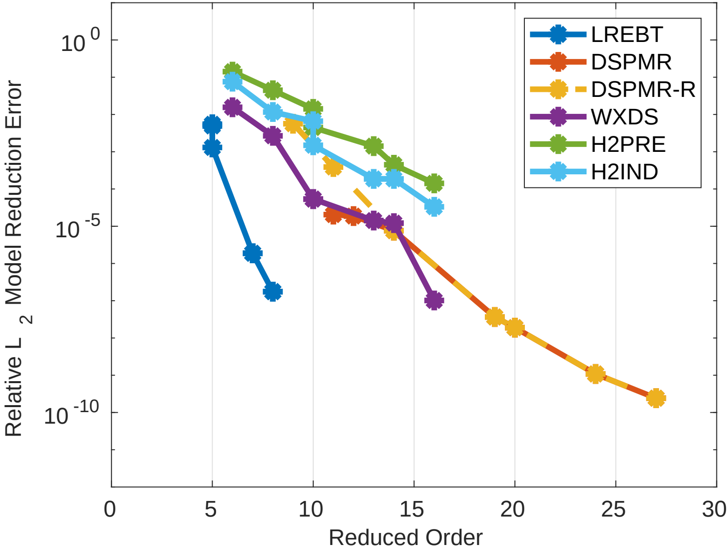
Projection Error vs Output Error $(v = \frac{1}{2})$



Reduced Order vs Output Error $(v = 0)$



Reduced Order vs Output Error $(v = \frac{1}{2})$



Linear Advection

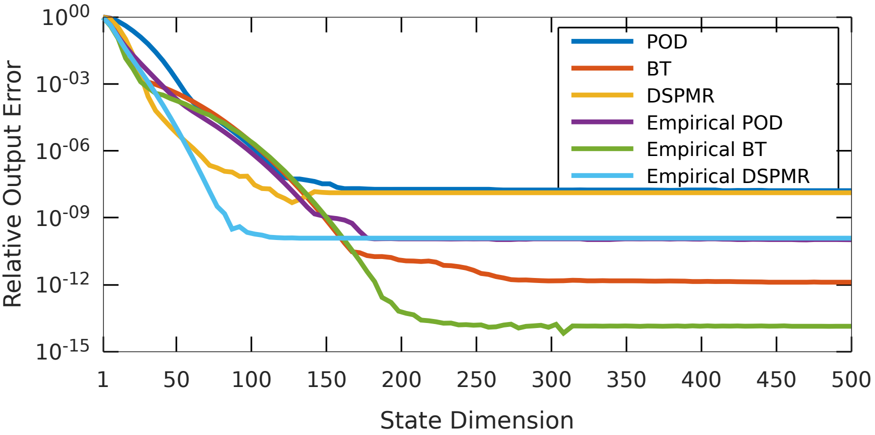
One Dimensional Linear Transport:

$$\begin{aligned} \frac{\partial z}{\partial t} &= -a \frac{\partial z}{\partial x} \\ z(0,t) &= u(t) \\ y(t) &= z(1,t), \end{aligned}$$

- Pure hyperbolic PDE
- Velocity a (fixed)
- Spatial finite difference (upwind) discretization **(SISO, N=1000)**:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

Reduced Order vs Output Error



S. Grundel, C.H., J. Saak. **On Empirical System Gramians**. Proceedings in Applied Mathematics and Mechanics 19: Accepted, 2019.

Gas Networks

Isothermal Euler Equations for Gas Flow in a Pipe:

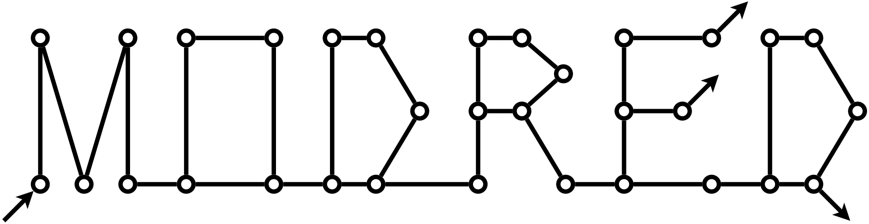
$$
\begin{aligned}
\frac{\partial}{\partial t}\rho &= -\frac{1}{S}\frac{\partial}{\partial x}q \\
\frac{\partial}{\partial t}q &= -S\frac{\partial}{\partial x}(R_S T z \rho) - S g \rho \frac{\partial}{\partial x}h - \frac{\lambda}{2DS}\frac{q|q|}{\rho} \\
p &= R_S T_0 z \rho
\end{aligned}
$$

Semi-Discrete Gas Network Model (Square MIMO):

$$
\begin{aligned}
\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} &= \begin{pmatrix} 0 & A_{pq} \\ A_{qp} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 & B_d \\ B_s & 0 \end{pmatrix} \begin{pmatrix} p_s \\ q_d \end{pmatrix} + \begin{pmatrix} 0 \\ f_q(p, q, p_s, q_d, \theta) \end{pmatrix} \\
\begin{pmatrix} p_d \\ q_s \end{pmatrix} &= \begin{pmatrix} C_d & 0 \\ 0 & C_s \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}
\end{aligned}
$$

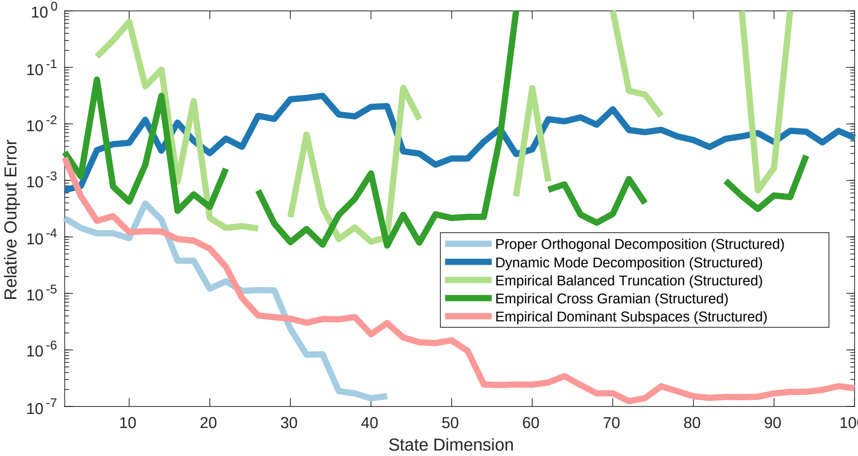
P. Benner, S. Grundel, C.H., C. Huck, T. Streubel, C. Tischendorf. **Gas Network Benchmark Models**. In: Differential-Algebraic Equation Forum: 171–197, 2018.

MODRED Network



- Cyclic graph
- Boundary nodes: 1 supply, 3 demand
- System dimensions: 4 inputs, **900 states**, 4 outputs
- Separate projections for pressure and mass-flux variables
- Generic training inputs

Reduced Order vs Output Error



Outlook

- Sharpen error indicator?
- Parameteric Systems (parametric error indicator?).
- Nonlinear Systems (Gas networks simulations)!

Summary

Empirical Dominant Subspaces:

- Concatenate controllability and observability subspace,
- obtained from the (empirical) cross Gramian,
- with an a-priori error indicator.

https://himpe.science

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emgr — EMpirical GRamian Framework (Version: 5.7)

Empirical Gramians:

- Empirical Reachability Gramian
- Empirical Observability Gramian
- Empirical Linear Cross Gramian
- Empirical Cross Gramian
- Empirical Sensitivity Gramian
- Empirical Identifiability Gramian
- Empirical Joint Gramian

Features:

- Open-source **OCTAVE** and **MATLAB** toolbox, **PYTHON** support.
- Interfaces for: Solver, inner product kernels & low-rank computation.
- Configurable, vectorized and parallelizable.

More info: https://gramian.de

C.H. emgr – **The Empirical Gramian Framework**. Algorithms 11(7): 91, 2018. doi:c9hj

Online Time (No Hyperreduction)