

- Empirical system Gramians
- generalize linear
- system-theoretic MOR
- to nonlinear and parametric
- input-output systems.
- emgr is an implementation.

## emgr – EMpirical GRamian Framework

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**Introduction**

**Abstract:** A wide range of industrial applications from control engineering, systems engineering, computational engineering to operations optimization are addressed by multivariable system theory. Including input-output modeling, the linear systems of interest are often nonlinear and subject to uncertainty. While standard linear systems may not suffice to model and analyze dynamics of complex technical systems, the use of nonlinear or parametric systems, for those cases empirical system Gramians provide the first step with the Empirical Gramian framework (emgr). It is an open source software for their computation.

**Applications:**

- Model Order Reduction
- Control Design
- Control Placement
- Sensitivity Analysis
- Structural Identification
- System Characterization

■ Parameter Reduction  
 ■ Combined Data and Parameter Reduction  
 ■ Nonlinear Quantification  
 ■ Uncertainty Quantification  
 ■ System Characterization

### Empirical System Gramians

Given a nonlinear (and parametric) input-output system, with vector field  $\dot{x}(t) = F(t, x(t), u(t)) + g(t, x(t), u(t), \theta)$

**Reachability – Input-To-State**  
 Perturb in  $i$ -th component of training input  $u_i$ , compute associated state trajectory  $x^{(i)}$ , and form average Gramian matrix:  

$$W_R = \frac{1}{2\pi} \sum_{i=1}^m \sum_{j=1}^m \int_0^T x_i^{(i)}(t) x_j^{(j)*}(t) dt$$
 → Empirical Reachability Gramian

**Observability – State-To-Output**  
 Perturb in  $i$ -th component of training initial state  $x_i$ , compute associated output trajectory  $y^{(i)}$ , and form average Gramian matrix:  

$$W_O = \frac{1}{2\pi} \sum_{i=1}^m \sum_{j=1}^m \int_0^T y_i^{(i)}(t) y_j^{(j)*}(t) dt$$
 → Empirical Observability Gramian

**Minimality – Input-To-Output**  
 Perturb in  $i$ -th component of training input  $u_i$  and compute associated state trajectory  $x^{(i)}$ , perturb in  $j$ -th component of training initial state  $x_j$  and compute output trajectory  $y^{(j)}$ , and form average cross-correlation matrix:  

$$W_{IO} = \frac{1}{2\pi} \sum_{i=1}^m \sum_{j=1}^m \int_0^T x_i^{(i)}(t) y_j^{(j)*}(t) dt$$
 → Empirical Cross Gramian

**Sensitivity – Parameter Reachability**  
 Treat  $i$ -th component of parameter  $\theta_i$  as an input, compute empirical reachability Gramian  $W_R(\theta_i)$ , and form diagonal matrix from trace:  

$$W_{SR} = \text{diag}(w_i)$$
 → Empirical Sensitivity Gramian

**Identifiability – Parameter Observability**  
 Treat parameter  $\theta_i$  as (combined) state, compute augmented empirical observability Gramian  $W_O(\theta_i)$ , and form Schur complement:  

$$W_I = W_O - W_R W_{SR} W_O$$
 → Empirical Identifiability Gramian

**Combined State Minimality and Parameter Observability**  
 Test parameter  $\theta_i$  as (combined) state, compute augmented empirical cross Gramian  $W_{IO}(\theta_i)$ , and form Schur complement of symmetric part:  

$$W_{CI} = \frac{1}{2} W_{IO} W_{SR} + W_{IO}^T W_I$$
 → Empirical Cross Identifiability Gramian

**KEYWORDS:**

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