Gramian-Based Model Reduction: From ABC-Systems to Gas Networks

Christian Himpe

AGO Seminar

2022–08–30

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› Outline

1. Pizza
2. Linear Systems
3. Model Reduction
4. Gas Networks
5. Numerical Results

Linear Systems

ABC-System

Linear Time-Invariant Input-Output System:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t)$$

- Input: $u : \mathbb{R} \to \mathbb{R}^M$
- State: $x : \mathbb{R} \to \mathbb{R}^N$
- Output: $y : \mathbb{R} \to \mathbb{R}^Q$
- Input Matrix: $B \in \mathbb{R}^{N \times M}$
- System Matrix: $A \in \mathbb{R}^{N \times N}$
- Output Matrix: $C \in \mathbb{R}^{Q \times N}$

› DEF-Extensions

Linear Time-Invariant Input-Output System with Feed-Forward:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

- Feed-Forward Matrix: $D \in \mathbb{R}^{M \times Q}$
- D is typically excluded from the reduction process.

› DEF-Extensions

Generalized Linear Time-Invariant Input-Output System:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

- Mass Matrix: $E \in \mathbb{R}^{N \times N}$
- If E is singular, this becomes a *descriptor system*.

› DEF-Extensions

Linear Time-Invariant Input-Output System with Source:

$$\dot{x}(t) = A\,x(t) + B\,u(t) + F$$
$$y(t) = C\,x(t)$$

- Source Vector: $F \in \mathbb{R}^N$
- Also known as *load vector*.

› DEF-Extensions

Linear Parametric Linear Time-Invariant Input-Output System:

$$\dot{x}(t) = A\,x(t) + B\,u(t) + F\theta$$
$$y(t) = C\,x(t)$$

- Source Matrix: $F \in \mathbb{R}^{N \times P}$
- Parameter: $\theta \in \mathbb{R}^{P}$

› DEF-Extensions

Parametric Linear Time-Invariant Input-Output System:

$$\dot{x}(t) = A(\theta)\, x(t) + B(\theta)\, u(t)$$
$$y(t) = C(\theta)\, x(t)$$

- System Map: $A : \mathbb{R}^P \to \mathbb{R}^{N \times N}$
- Input Map: $B : \mathbb{R}^P \to \mathbb{R}^{N \times M}$
- Output Map: $C : \mathbb{R}^P \to \mathbb{R}^{Q \times N}$
- Parameter: $\theta \in \mathbb{R}^P$

› DEF-Extensions

Affine Parametric Linear Time-Invariant Input-Output System:

$$\dot{x}(t) = \left(A_0 + \sum_{k=1}^{K} A_k(\theta)\right) x(t) + B\,u(t)$$
$$y(t) = C x(t)$$

- System Maps: $A_k : \mathbb{R}^P \to \mathbb{R}^{N \times N}$
- Parameter: $\theta \in \mathbb{R}^P$

› Port-Hamiltonian Systems

Linear Time-Invariant Port-Hamiltonian Input-Output System:

$$
\begin{aligned}
E\dot{x}(t) &= \overbrace{(J-R)Q}^{A}\,x(t) + \overbrace{(G-P)}^{B}\,u(t) \\
y(t) &= \underbrace{(G+P)^\top Q}_{C}\,x(t)
\end{aligned}
$$

- Energy Flux: $J = -J^\top$
- Energy Dissipation: $R = R^\top \geq 0$
- Energy Storage: $Q = Q^\top > 0$
- Mass Matrix: $E = E^\top > 0$
- Control Ports: G
- Resistive Ports: P

Model Reduction

› Projection-Based Model Reduction

Reconstructing and Reducing Projections:

$$U_n \in \mathbb{R}^{N \times n}, \quad V_n \in \mathbb{R}^{n \times N}, \quad VU = \mathbb{1}$$

Reduced Order State:

$$x_r(t) := V_n x(t) \in \mathbb{R}^n \rightarrow x(t) \approx U_n x_r(t) \in \mathbb{R}^N$$

Reduced Order Model:

$$\begin{aligned} U_n V_n \dot{x}(t) &= A U_n V_n x(t) + B u(t) \\ \tilde{y}(t) &= C U_n V_n x(t) \end{aligned}$$

› Projection-Based Model Reduction

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Reduced Order Model:

$$\begin{aligned} \dot{x}_r(t) &= (V_n A U_n) x_r(t) + (V_n B) u(t) \\ \tilde{y}(t) &= (C U_n) x_r(t) \end{aligned}$$

› Projection-Based Model Reduction

Reconstructing and Reducing Projections:

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Reduced Order State:

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Reduced Order Model:

$$\begin{aligned} \dot{x}_r(t) &= A_r x(t) + B_r u(t) \\ \tilde{y}(t) &= C_r x(t) \end{aligned}$$

› Controllability Gramians

Controllability Gramian:

$$W_C := \int_0^\infty e^{A x(t)} B B^\top e^{A^\top x(t)} \,\mathrm{d}t$$

- Controllability: Ability to drive a system to a steady-state.
- Impulse Response: $e^{A x(t)} B = \left(B^\top e^{A^\top x(t)}\right)^\top$.
- Lyapunov equation: $A W_C + W_C A^\top = -B B^\top$.

› Proper Orthogonal Decomposition

System-Theoretic View of POD:

$$W_C \overset{\text{SVD}}{=} U \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} U^\top$$

Energy-fraction-based truncation:

$$\frac{\sum_{k=1}^{n} \lambda_k}{\sum_{k=1}^{N} \lambda_k} > (1 - \varepsilon) \rightarrow U_n := U \begin{bmatrix} \mathbb{1}_n \\ 0_{N-n} \end{bmatrix} \rightarrow V_n = U_n^\top$$

› Goal-Oriented POD

Output Controllability Gramian:

$$CW_C C^\top \overset{\text{SVD}}{=} CU \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} U^\top C^\top = \hat{C} \Lambda \hat{C}^\top$$

Sort by Impulse-response norm:

$$\|y\|_2 = \sqrt{\operatorname{tr}(C\, W_C\, C^\top)} = \sqrt{\sum_{k=1}^{N} (\hat{c}_{k,*}\, \hat{c}_{k,*}^\top\, \lambda_k)}$$

› Observability Gramians

Observability Gramian:

$$W_O := \int_0^\infty e^{A^\top x(t)} C^\top C e^{A x(t)} \, \mathrm{d}t$$

- Observability: Ability to see changes in states in outputs.
- Adjoint Impulse Response: $e^{A^\top x(t)} C^\top = (C e^{A x(t)})^\top$.
- Lyapunov equation: $A W_O + W_O A^\top = -C^\top C$.

› Modified POD

POD and Adjoint POD:

$$W_C \overset{\text{SVD}}{=} U \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} U^\top, \qquad W_O \overset{\text{SVD}}{=} V \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_N \end{pmatrix} V^\top$$

Oblique Projection ($VU \neq \mathbb{1}$):

$$V_n := \begin{bmatrix} \mathbb{1}_n & 0_{N-n} \end{bmatrix} V, \quad U_n := U \begin{bmatrix} \mathbb{1}_n \\ 0_{N-n} \end{bmatrix}$$

› Dominant Subspace Projection Model Reduction

POD and Adjoint POD:

$$W_C \overset{\text{SVD}}{=} U \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} U^\top, \qquad W_O \overset{\text{SVD}}{=} V \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_N \end{pmatrix} V^\top$$

Galerkin Projection:

$$U_n \Sigma_n \tilde{V}_n \overset{\text{SVDs(n)}}{=} \left[\left(\begin{bmatrix} \mathbb{1}_n & 0_{N-n} \end{bmatrix} V \right) \quad \left(U \begin{bmatrix} \mathbb{1}_n \\ 0_{N-n} \end{bmatrix} \right) \right] \to V_n = U_n^\top$$

› Balanced Truncation

Balancing Transformation:

$$U \Sigma V \overset{SVD}{=} W_O W_C$$

Petrov-Galerkin Projection:

$$V_n := \begin{bmatrix} \mathbb{1}_n & 0_{N-n} \end{bmatrix} \Sigma^{-1} U^\top W_O, \quad U_n := W_C V \Sigma^{-1} \begin{bmatrix} \mathbb{1}_n \\ 0_{N-n} \end{bmatrix}$$

Balanced Gains

Balancing Transformation:

$$U \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N \end{pmatrix} V \overset{SVD}{=} W_O W_C$$

Sort by Impulse-response norm:

$$\|y\|_2 = \sqrt{\operatorname{tr}(\hat{C} \widehat{W}_C \hat{C}^\top)} = \sqrt{\sum_{k=1}^{N} \hat{c}_{k,*} \hat{c}_{k,*}^\top \sigma_k}$$

Balanced POD

Approximate Balancing Transformation:

$$W_O \overset{SVDs}{=} U_O \, M \, U_O^T, \qquad W_C \overset{SVDs}{=} V_C \, \Lambda \, V_C^T$$

$$U \, \Sigma \, V \overset{SVD}{=} U_O \, V_C^\top$$

Petrov-Galerkin Projection:

$$V_n := \begin{bmatrix} \mathbb{1}_n & 0_{N-n} \end{bmatrix} \Sigma^{-1} U^\top U_O, \qquad U_n := V_C V \Sigma^{-1} \begin{bmatrix} \mathbb{1}_n \\ 0_{N-n} \end{bmatrix}$$

› Cross Gramians

Cross Gramian:

$$W_X := \int_0^\infty \mathrm{e}^{A\,x(t)}\, B\, C\, \mathrm{e}^{A\,x(t)}\, \mathrm{d}t$$

- Minimality: How many states are minimally required.
- Cross Operator: $(\mathrm{e}^{A\,x(t)}\, B)(C\,\mathrm{e}^{A\,x(t)})$.
- Sylvester equation: $A\,W_X + W_X\,A = -B\,C$.

› DMD-Galerkin

Dynamic Mode Decomposition (DMD) in 1 minute

1. $X = \begin{bmatrix} x_0 & \dots & x_K \end{bmatrix}$
2. $X_0 := \begin{bmatrix} x_0 & \dots & x_{K-1} \end{bmatrix}, \quad X_1 := \begin{bmatrix} x_1 & \dots & x_K \end{bmatrix}$
3. $x_{k+1} \stackrel{!}{=} A\, x_k$
4. $X_1 = A_{\text{DMD}}\, X_0$
5. $A_{\text{DMD}} = X_1\, X_0^+$

DMD-Galerkin (Empirical Modal Truncation):

$$A_{\text{DMD}} \stackrel{\text{SVDs}}{=} U_n\, \Lambda_n\, \tilde{V}_n \to V_n = U_n^\top$$

› Nonlinear Systems

(Adjoint) Impulse Response:

- $x(t) = \mathrm{e}^{A\,x(t)}\,B\,u(t)$
- $z(t) = \mathrm{e}^{A^\top x(t)}\,C^\top v(t)$

Empirical Gramians:

- Empirical Controllability Gramian: $\widehat{W}_C = \int_0^\infty x(t)\,x(t)^\top \,\mathrm{d}t$
- Empirical Observability Gramian: $\widehat{W}_O = \int_0^\infty z(t)\,z(t)^\top \,\mathrm{d}t$
- Empirical Cross Gramian: $\widehat{W}_X = \int_0^\infty x(t)\,z(t)^\top \,\mathrm{d}t$

(Nonlinear systems use $x(t)$ and $y(t)$)

› Parametric Systems

Parametric Input-Output System:

$$\dot{x}(t) = f(x(t), u(t), \theta, t)$$
$$y(t) = g(x(t), u(t), \theta, t)$$

Average Gramian:

$$\overline{W}_* = \sum_{p=1}^{P} W_*(\theta_p)$$

(Applies also to time-varying systems)

› Structured Systems

(Block) Structured Linear Time-Invariant System:

$$\begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} A_{pp} & A_{pq} \\ A_{qp} & A_{qq} \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} + \begin{pmatrix} B_p \\ B_q \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} C_p & C_q \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

(Block) Structured System Gramians:

$$W_* = \begin{pmatrix} W_{*,pp} & W_{*,pq} \\ W_{*,qp} & W_{*,qq} \end{pmatrix}$$

› Hyperbolic Systems

Empirical Gramians for Hyperbolic Systems:

- (Linear) system theory is centered around impulse responses.
- Transport can introduce a delay between input and outputs.
- Impulse input may dissipate before output is reached.
- Step input is an alternative.
- But: (Infinite-time) Gramians are not defined for step inputs.
- However: Step response Gramians are defined on finite time.
- All empirical Gramians are practically time-limited.

› Gain Matching

System Gain:

$$S = h(0) = C(\mathbb{1} \overbrace{s}^{=0} - A)^{-1} B = C A^{-1} B$$

Gain Matching by adding feed-through to ROM:

$$D_r := C A^{-1} B - C_r A_r^{-1} B_r$$

› Gain Matching

System Gain:

$$S = h(0) = C(\mathbb{1} \overbrace{s}^{=0} - A)^{-1} B = C A^{-1} B$$

Gain Matching by adding feed-through to ROM (port-Hamiltonian FOM):

$$D_r := C Q^{-1} B - C_r Q_r^{-1} B_r$$

Gas Networks

› Gas Pipeline

Friction-Dominated 1D Isothermal Euler Equations:

$$\frac{1}{T_0\, R_S\, z_0}\,\partial_t p = -\frac{1}{S}\,\partial_x q$$

$$\partial_t q = -S\,\partial_x p - \Big(\overbrace{\frac{S\, g\, h_x}{T_0\, R_S\, z_0}}^{\text{Gravity}} p + \overbrace{\frac{T_0\, R_S\, z_0 \lambda_0}{2\, d\, S} \frac{|q|\, q}{p}}^{\text{Friction}} \Big)$$

- Pressure: $p(x,t)$
- Mass-Flux: $q(x,t)$
- Pipe Incline: h_x
- Pipe Diameter: d
- Pipe Cross-Section: S
- Friction Factor: λ_0
- Compressibility Factor: z_0
- Temperature: $T_0 =: \theta_1$
- Specific Gas Constant: $R_S =: \theta_2$
- Gravity Acceleration: g

› Gas Networks

Kirchhoff Laws:

1. The sum of mass-flux in- and outflows at every junction is zero:

$$\mathcal{A}\, q(t) = \mathcal{B}_d\, d_q(t)$$

2. The sum of pressure drops in every fundamental loop is zero.
   $\rightarrow$ Nodal pressures at in-flows equal to boundary function:

$$\mathcal{A}_0^\top\, p(t) + \mathcal{B}_s^\top\, s_p(t) = |\mathcal{A}_0^\top|\, p(t)$$

- Incidence Matrix: $\mathcal{A}$
- Reduced Incidence Matrix: $\mathcal{A}_0$
- Boundary Pressure Map: $\mathcal{B}_s$
- Boundary Mass-Flux Map: $\mathcal{B}_d$
- Pressure: $p(t)$
- Mass-Flux: $q(t)$
- Pressure Boundary: $s_p(t)$
- Mass-Flux Boundary: $d_q(t)$

› Input-Output System

Square Input-Output-System (ODE) with Compressors:

$$\begin{pmatrix} E_p(\theta) & 0 \\ 0 & E_q \end{pmatrix} \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & A_{pq} \\ \hat{A}_{qp} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 & B_{pd} \\ B_{qs} & 0 \end{pmatrix} \begin{pmatrix} s_p \\ d_q \end{pmatrix} + \begin{pmatrix} 0 \\ F_C + f_q(p, q, \theta) \end{pmatrix}$$

$$\begin{pmatrix} s_q \\ d_p \end{pmatrix} = \begin{pmatrix} 0 & C_{sq} \\ C_{dp} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

Input:

- Pressure @ supply: s_p
- Mass-flux @ demand: d_q

State:

- Pressure: p
- Mass-flux: q

Output:

- Pressure @ demand: d_p
- Mass-flux @ supply: s_q

› Practical Remarks

Gas Networks:

- Compressors $\rightarrow$ Source term
- Initial values $\rightarrow$ Steady-State problem
- Exchangeable compressibility factor formula
- Exchangeable friction factor formula
- Tuning factor to match real data

Model Reduction:

- Structuring, nonlinearity, parametricity voids BT properties
- Hyperbolicity may require large reduced order model (ROM)
- Galerkin is stability preserving for port-Hamiltonian systems
- Approximate adjoint for local repeated scalar nonlinearities
- Quality of ROM depends on quality of data (and thus solver)

Numerical Experiments

› MORGEN - Model Order Reduction for Gas and Energy Networks

Design Principles:

- No Optimization (Steady-State, Time-Stepper, ROMs, Stabilization)
- Approximate Adjoint if available (Port-Hamiltonian + SRSN)
- Only global, but structured projectors (Unstructured $\rightarrow$ Unstable)

General Features:

- Open-Source
- MATLAB & Octave Compatibility
- Modular / Configurable / Extensible

Specific Features:

- Holistic Approach: Model–Solver–Reductor Ensembles
- Data-Driven System-Theoretic MOR
- Realistic Test Networks

› Experimental Setup

Training:

- Short Offline phase (1-12h, depending on network expanse).
- Generic training scenarios (step inputs).
- Sparse grid parameter sampling.

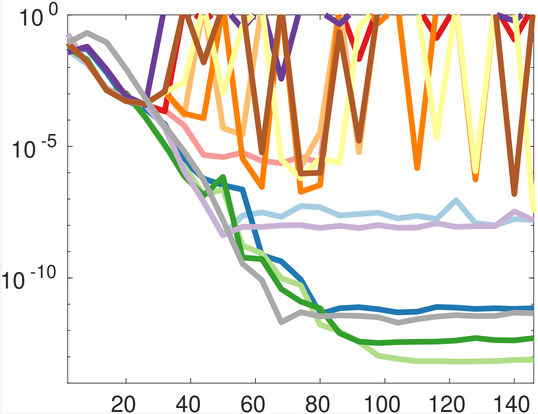
Testing:

- Long testing phase (24h).
- Designed, random or realistic scenarios.
- Uniformly distributed parameter sampling.

Evaluation:

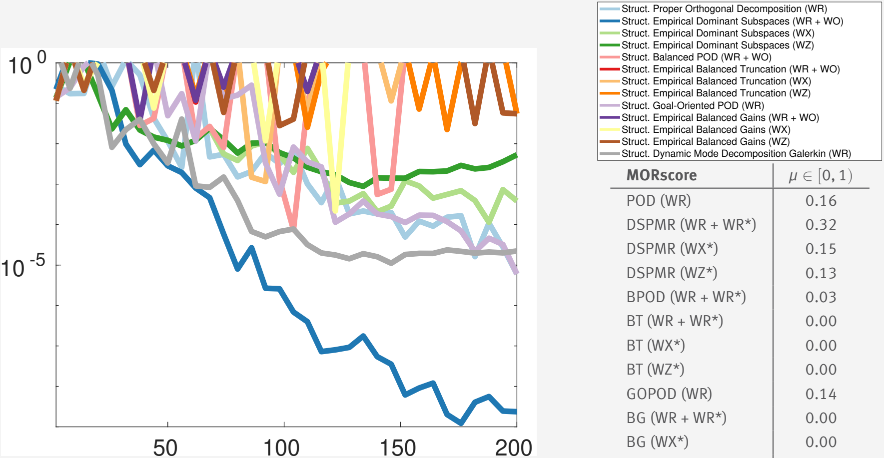
- $L_2 \otimes L_2$-Norm model reduction error (also $L_x \otimes L_y; x, y = 0, 1, 2, \infty$)
- MORscore: Area above reduced-order-vs-error graph.
- Averaged over multiple parameter samples.

› Yamal-Europe Pipeline $(N \approx 1000)$



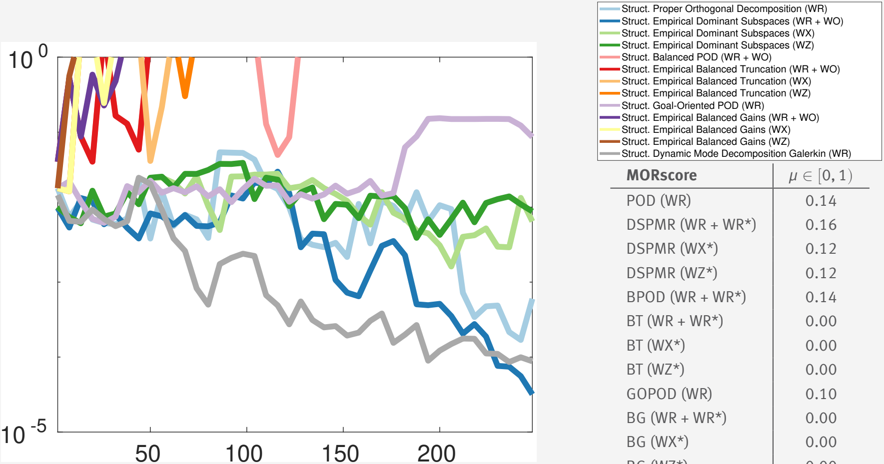
MORscore	$\mu \in [0, 1)$
POD (WR)	0.40
DSPMR (WR + WR*)	0.51
DSPMR (WX*)	0.57
DSPMR (WZ*)	0.56
BPOD (WR + WR*)	0.14
BT (WR + WR*)	0.03
BT (WX*)	0.10
BT (WZ*)	0.15
GOPOD (WR)	0.41
BG (WR + WR*)	0.02
BG (WX*)	0.14
BG (WZ*)	0.11
DMD-G (WR)	0.53

› MORGEN-Network ($N \approx 1000$)



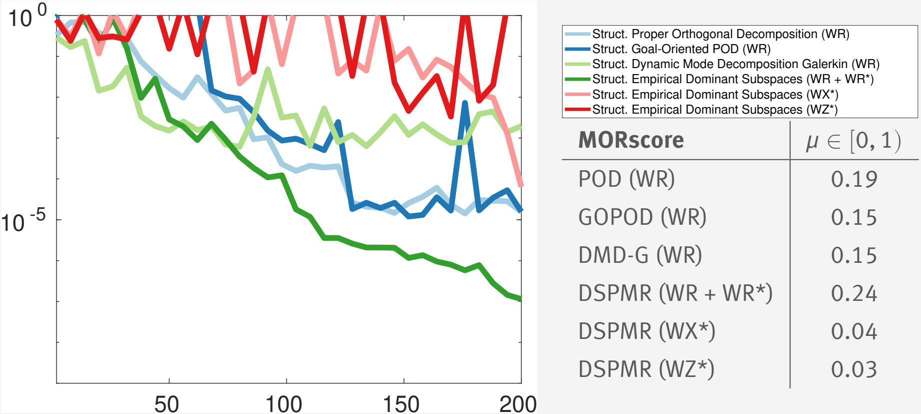
MORscore	$\mu \in [0, 1)$
POD (WR)	0.16
DSPMR (WR + WR*)	0.32
DSPMR (WX*)	0.15
DSPMR (WZ*)	0.13
BPOD (WR + WR*)	0.03
BT (WR + WR*)	0.00
BT (WX*)	0.00
BT (WZ*)	0.00
GOPOD (WR)	0.14
BG (WR + WR*)	0.00
BG (WX*)	0.00
BG (WZ*)	0.01
DMD-G (WR)	0.21

› GasLib-134 $(N \approx 2700)$



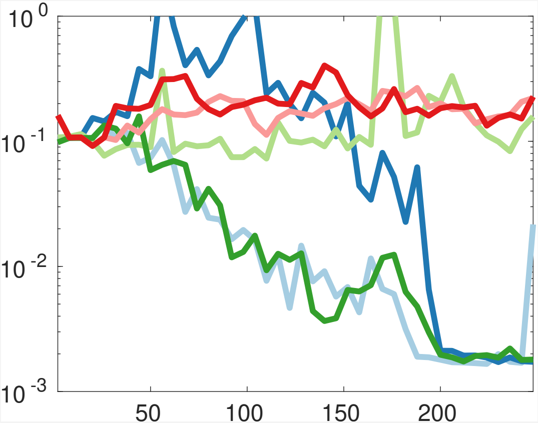
MORscore	$\mu \in [0, 1)$
POD (WR)	0.14
DSPMR (WR + WR*)	0.16
DSPMR (WX*)	0.12
DSPMR (WZ*)	0.12
BPOD (WR + WR*)	0.14
BT (WR + WR*)	0.00
BT (WX*)	0.00
BT (WZ*)	0.00
GOPOD (WR)	0.10
BG (WR + WR*)	0.00
BG (WX*)	0.00
BG (WZ*)	0.00
DMD-G (WR)	0.19

› American Gas Association Net ($N \approx 1500$)



MORscore	$\mu \in [0, 1)$
POD (WR)	0.19
GOPOD (WR)	0.15
DMD-G (WR)	0.15
DSPMR (WR + WR*)	0.24
DSPMR (WX*)	0.04
DSPMR (WZ*)	0.03

› Chinese Transport Network ($N \approx 11000$)



Struct. Proper Orthogonal Decomposition (WR)
Struct. Goal-Oriented POD (WR)
Struct. Dynamic Mode Decomposition Galerkin (WR)
Struct. Empirical Dominant Subspaces (WR + WR*)
Struct. Empirical Dominant Subspaces (WX*)
Struct. Empirical Dominant Subspaces (WZ*)

MORscore	$\mu \in [0, 1)$
POD (WR)	0.12
GOPOD (WR)	0.07
DMD-G (WR)	0.06
DSPMR (WR + WR*)	0.12
DSPMR (WX*)	0.05
DSPMR (WZ*)	0.04

› Summary

Findings:

- Model: **Port-Hamiltonian Endpoint Discretization**
- Solver: **First-Order Implicit-Explicit**
- Reductor: **Dominant Subspace Projection Model Reduction**

https://himpe.science

https://gramian.de

https://git.io/morgen

https://git.io/hapod

christian.himpe@wwu.de

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